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Periodic Travelling Waves of Forced FPU Lattices

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Abstract In this article, damped Fermi–Pasta–Ulam-type lattices driven by extended external forces are considered. The existence and uniqueness results of periodic travelling waves of the system are presented. The existence and the stability of periodic waves are also computed and discussed numerically.

Keywords Travelling waves · Lattice waves · Forced FPU lattice

1 Introduction

The Fermi–Pasta–Ulam (FPU) model formulated in an attempt to explain heat conduction in non-metallic lattices [17] became a cornerstone in modern statistical mechanics [20,30].

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Since then, the anharmonic oscillators have been used to study various problems, such as the relation between stochastic motions and thermodynamics properties [24,33] and the heat conduction in a 1D chain [23,29].

The absence of energy equipartition in the model, which was paradoxical at the time [3], has stimulated the study of nonlinear dynamics and chaos [34]. Perhaps the best-known explanation to date of the FPU recurrence is the localization in the normal-mode (Fourier) space of the so-called q-breathers [9,19]. The near-recurrence phenomenon was also studied in the continuum limit by Zabusky and Kruskal [44], who derived the integrable Korteweg-de Vries equation that admits spatial energy-localization. Note that in the high-frequency limit, one can approximate the FPU model with the nonlinear Schrödinger equation admitting similar localized excitations [4,30]. Takeno, Kisoda and Sievers [40–43] did observe spatially localized vibrational excitations in the same FPU models, which were unexpected since the excitations violate the underlying discrete translational symmetry of the model. Such localizations are later referred to as discrete breathers or intrinsic localized modes [18]. Discrete breathers are also obtained as a transient state due to modulational instabilities when initially all the energy was fed in the highest frequency mode [7,8,39] above a critical energy that was analytically derived in [11]. Only recently the (non)existence of small amplitude breathers in FPU lattices was proved analytically using a centre manifold reduction method when the coupling potential (dis)satisfies a local hardening condition [22] (see [38] for comparisons of the analytical results and numerics).

Discrete breathers of FPU models can also be created by applying a sinusoidal drive at one edge of a semi-infinite chain [27]. An amplitude drive exceeding a critical value will trigger the creation of moving excitations. Periodic drives across lattices have been proposed earlier and studied in [35–37] motivated by the question whether discrete breathers can occur in the presence of a spatially extended external driving force. A critical value of driving amplitude above which discrete breathers are created was later calculated in [25]. The case of a general wavenumber along the lattice sites was considered in [26]. Here, we extend and complement the results of [25,26] in the analysis of the existence and uniqueness of periodic travelling waves of forced-damped FPU type lattices (see in particular Remark 1 for the details). In a moving coordinate frame, the model system becomes an advanced-delay differential equation.

This paper is a continuation of [14,16,15] on the existence of periodic travelling waves in Hamiltonian lattices, such as discrete Klein-Gordon equations and nonlocal discrete nonlinear Schrödinger equations in one-dimension and higher as well, where methods of dynamical systems and nonlinear functional analysis are employed. The aim of this paper is to answer a particular question on the influence of forcing that may be present due to an external field and dissipation to the existence and stability of waves in FPU systems (see, e.g., [32] for a review of the undamped, undriven case). A similar problem is studied in [12] for time-discretized damped and forced FPU lattices.

The paper is outlined as follows. In Sect. 2, we introduce the governing equation and the analytical setup to prove the existence and uniqueness of periodic travelling wave in the system. An analytical expression of driving amplitude that limits our analysis is derived in the section. A resonance condition is also discussed. In Sect. 3, we extend the analysis to the case of general nonlocal interactions between the lattice sites. In Sect. 4, the resonance condition discussed in Sect. 2 is revisited and the existence of periodic solutions is proven using Lyapunov–Schmidt reduction method. Also some comments about results of [25,26] and the comparison with ours are presented. Finally, we solve the advance-delay equation numerically using a pseudo-spectral method and establish the stability of periodic solutions by numerically calculating the corresponding Floquet multipliers in Sect. 5.

2 1D Forced FPU Lattices with Local Interactions

We consider a 1D damped FPU lattice forced by a travelling wave field (see [25,26]):

$$\begin{aligned}\ddot{u}_n &= \alpha\,(u_{n+1}+u_{n-1}-2u_n)+\beta(u_{n+1}-u_n)^3+\beta(u_{n-1}-u_n)^3\\ &\quad -\gamma\dot{u}_n+f\cos(\omega t+pn),\end{aligned} \tag{1}$$

where $\alpha>0$, $\beta>0$, $\gamma\geq 0$, $\omega>0$, $p\neq 0$, $f\neq 0$ are parameters.

Putting $u_n(t)=U(\omega t+pn)$ in (1), we obtain

$$\begin{aligned}\omega^2U''(z) &= \alpha\,(U(z+p)+U(z-p)-2U(z))+\beta(U(z+p)-U(z))^3\\ &\quad +\beta(U(z-p)-U(z))^3-\gamma\omega U'(z)+f\cos z,\end{aligned} \tag{2}$$

and seek solutions of the advance-delay equation satisfying the property

$$U(z+\pi)=-U(z). \tag{3}$$

We take Banach spaces

$$\begin{aligned}W &:= \left\{U\in C(\mathbb{R},\mathbb{R})\mid U(z)=\sum_{k\in\mathbb{Z}}d_k\mathrm{e}^{k\imath z},\ \sum_{k\in\mathbb{Z}}|d_k|<\infty\right\},\\ X &:= \left\{U\in C(\mathbb{R},\mathbb{R})\mid U(z)=\sum_{k\in\mathbb{Z}}c_k\mathrm{e}^{(2k+1)\imath z},\ \sum_{k\in\mathbb{Z}}|c_k|<\infty\right\},\\ Y &:= \left\{U\in C^1(\mathbb{R},\mathbb{R})\mid U(z)=\sum_{k\in\mathbb{Z}}c_k\mathrm{e}^{(2k+1)\imath z},\ \sum_{k\in\mathbb{Z}}|2k+1||c_k|<\infty\right\},\\ Z &:= \left\{U\in C^2(\mathbb{R},\mathbb{R})\mid U(z)=\sum_{k\in\mathbb{Z}}c_k\mathrm{e}^{(2k+1)\imath z},\ \sum_{k\in\mathbb{Z}}(2k+1)^2|c_k|<\infty\right\}\end{aligned}$$

with the norms

$$\begin{aligned}\|U\| &:= \sum_{k\in\mathbb{Z}}|d_k|,\quad \|U\|:=\sum_{k\in\mathbb{Z}}|c_k|,\\ \|U\|_1 &:= \sum_{k\in\mathbb{Z}}|2k+1||c_k|,\quad \|U\|_2:=\sum_{k\in\mathbb{Z}}(2k+1)^2|c_k|,\end{aligned}$$

respectively. It is easy to verify that $Z\circlearrowleft Y\circlearrowleft X$ are compact embeddings, $X\subset W$ and $\|U\|\leq\|U\|_1\forall U\in Y$, $\|U\|_1\leq\|U\|_2\forall U\in Z$.

The following lemma is clear.

Lemma 1 *If $U_1,U_2\in W$ then $U_1U_2\in W$ and $\|U_1U_2\|\leq\|U_1\|\|U_2\|$. If $U_1,U_2,U_3\in X$ then $U_1U_2U_3\in X$.*

By setting

$$\begin{aligned}\mathcal{K}U &:= \omega^2U''(z)+\gamma\omega U'(z)-\alpha U(z+p)-\alpha U(z-p)+2\alpha U(z),\\ \mathcal{F}(U,f) &:= \beta(U(z+p)-U(z))^3+\beta(U(z-p)-U(z))^3+f\cos z,\end{aligned}$$

(2) has the form

$$\mathcal{K}U=\mathcal{F}(U,f).$$

We have the next result.

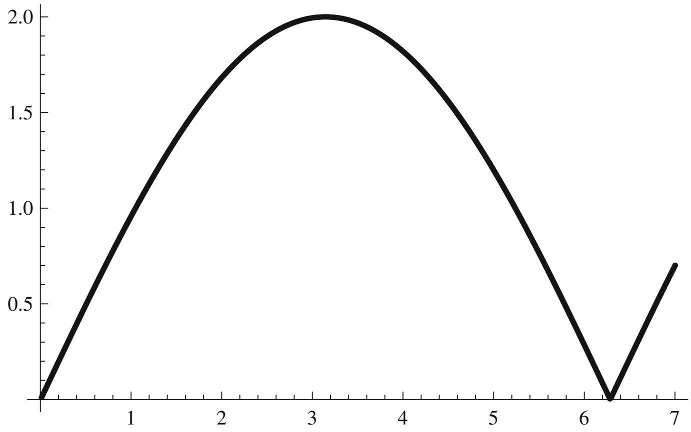

Fig. 1 The graph of function $\Delta(p)$ defined by (7) on interval [0, 7]

Lemma 2 *$\mathcal{F} : Y \times \mathbb{R} \to X$ with the properties*

$$\|\mathcal{F}(U, f)\| \leq 2\beta\Delta(p)^3\|U\|_1^3 + |f|, \tag{4}$$

$$\|\mathcal{F}(U_1, f)-\mathcal{F}(U_2, f)\| \leq 2\beta\Delta(p)^3\|U_1 - U_2\|_1 \left(\|U_1\|_1^2 + \|U_1\|_1\|U_2\|_1 + \|U_2\|_1^2\right), \tag{5}$$

$$\|\mathcal{F}(U, f_1) - \mathcal{F}(U, f_2)\| \leq |f_1 - f_2| \tag{6}$$

for any $U, U_1, U_2 \in Y$ and $f, f_1, f_2 \in \mathbb{R}$, where

$$\Delta(p) := 2 \max_{k\in\mathbb{Z}} \left|\frac{\sin \frac{2k+1}{2} p}{2k + 1}\right|. \tag{7}$$

Note that $\Delta \in C(\mathbb{R}, \mathbb{R})$ with the following properties

$$\begin{aligned} &0 \leq \Delta(p) \leq 2, \quad \Delta(p + 2\pi) = \Delta(p),\\ &\Delta(-p) = \Delta(p), \quad \Delta(p) \leq |p| \end{aligned} \tag{8}$$

for any $p \in \mathbb{R}$ (see Fig. 1).

Proof Since $\cos z = \frac{e^{\imath z}+e^{-\imath z}}{2}$, we obtain $\|\cos z\| = 1$ and (6) easily follows. Next we derive

$$U(z + p) - U(z) = \sum_{k\in\mathbb{Z}} c_k \left(e^{(2k+1)p\imath} - 1\right) e^{(2k+1)\imath z}$$

for any $U \in X$. So we obtain

$$\begin{aligned} \|U(z + p) - U(z)\| &= \sum_{k\in\mathbb{Z}} |c_k| \left|e^{(2k+1)p\imath} - 1\right| \\ &= \sum_{k\in\mathbb{Z}} |c_k|\sqrt{(\cos(2k + 1)p - 1)^2 + \sin^2(2k + 1)p} \\ &= 2\sum_{k\in\mathbb{Z}} |c_k| \left|\sin \frac{2k + 1}{2} p\right| \leq \Delta(p)\|U\|_1 \end{aligned} \tag{9}$$

for any $U \in Y$. By using (9) and Lemma 1, we arrive at (4).

Next arguing like for (9), we obtain

$$\|U_1(z+p) - U_1(z) - (U_2(z+p) - U_2(z))\| \leq \Delta(p)\|U_1 - U_2\|_1 \tag{10}$$

for any $U_1, U_2 \in Y$. Then property (5) follows immediately from Lemma 1, (9), (10) and a formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ for any $a, b \in \mathbb{C}$. The proof is finished.

Next if $U \in Z$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{(2k+1)\imath z}$ then

$$\mathcal{K}U(z) = \sum_{k\in\mathbb{Z}} \left(4\alpha \sin^2 \frac{2k+1}{2} p - (2k+1)^2\omega^2 + \gamma\omega(2k+1)\imath\right) c_k \mathrm{e}^{(2k+1)\imath z} \tag{11}$$

and so $\mathcal{K} \in L(Z, X)$ with

$$\|\mathcal{K}\|_{L(Z,X)} \leq 4\alpha + \omega^2 + \gamma\omega.$$

If

$$\alpha \frac{4\sin^2 \frac{2k+1}{2} p}{(2k+1)^2} - \omega^2 + \frac{\gamma\omega}{2k+1}\imath \neq 0 \quad \forall k \in \mathbb{Z}, \tag{12}$$

then

$$\Theta := \min_{k\in\mathbb{Z}} \sqrt{\left(4\alpha \frac{\sin^2 \frac{2k+1}{2} p}{2k+1} - (2k+1)\omega^2\right)^2 + \gamma^2\omega^2} > 0$$

for a constant Θ depending on $\alpha, p, \omega, \gamma$. Then we also have $\mathcal{K}^{-1} \in L(X, Z) \subset L(X, Y) \subset L(X)$, so $\mathcal{K}^{-1} : X \to Y$ is compact such that

$$\|\mathcal{K}^{-1}\|_{L(X,Y)} \leq \frac{1}{\Theta}. \tag{13}$$

Now we can prove the following existence results on (2) when all parameters except f are fixed.

Theorem 1 *Assume* (12) *along with*

$$|f| < \sqrt{\frac{2\Theta^3}{27\Delta(p)^3\beta}}. \tag{14}$$

Then (2) *has a unique solution $U(f) \in B(\rho_f)$ in a closed ball*

$$B(\rho_f) := \left\{U \in Y \mid \|U\|_1 \leq \rho_f\right\},$$

where

$$\rho_f = 2\sqrt{\frac{\Theta}{6\Delta(p)^3\beta}} \sin\left(\frac{1}{3} \arcsin\left(|f|\sqrt{\frac{27\Delta(p)^3\beta}{2\Theta^3}}\right)\right). \tag{15}$$

Moreover, $U(f)$ can be approximated by an iteration process. Finally, it holds

$$\|U(f_1) - U(f_2)\|_1 \leq \frac{|f_1 - f_2|}{\Theta\left(1 - 4\sin^2\left(\frac{1}{3}\arcsin\left(\max\{|f_1|, |f_2|\}\sqrt{\frac{27\Delta(p)^3\beta}{2\Theta^3}}\right)\right)\right)} \tag{16}$$

for any $f_1, f_2 \in \mathbb{R}$ satisfying (14).

Proof We rewrite (2) as a parameterized fixed point problem

$$U = \mathcal{R}(U, f) := \mathcal{K}^{-1}\mathcal{F}(U, f)$$

on Y. We already know that $\mathcal{R} : Y \times \mathbb{R} \to Y$ is compact, continuous and by (4) and (13) such that

$$\|\mathcal{R}(U, f)\|_1 \le \frac{1}{\Theta}\left(2\Delta(p)^3\beta\|U\|_1^3 + |f|\right).$$

Next, if there is a $\rho_f > 0$ such that

$$2\Delta(p)^3\beta\rho_f^3 + |f| = \Theta\rho_f, \tag{17}$$

then $\mathcal{R}(\cdot, f)$ maps a closed ball $B(\rho_f)$ into itself. So it remains to study (17). In order to find the largest f_l for which (17) has a solution $\rho_f > 0$, we need to solve

$$2\Delta(p)^3\beta\rho_1^3 + |f_l| = \Theta\rho_1, \quad 6\Delta(p)^3\beta\rho_1^2 = \Theta,$$

which gives $|f_l| = \sqrt{\frac{2\Theta^3}{27\Delta(p)^3\beta}}$. This implies (14). So assuming (12) and (14), we already know that (17) has a positive solution ρ_f. We take the smallest one, which is known to have a form of (15) [5]. Then $\mathcal{R}(\cdot, f)$ maps $B(\rho_f)$ into itself. Moreover, by (5) and (13)

$$\|\mathcal{R}(U_1, f) - \mathcal{R}(U_2, f)\|_1 \le \frac{6\Delta(p)^3\beta}{\Theta}\rho_f^2\|U_1 - U_2\|_1$$

for any $U_1, U_2 \in B(\rho_f)$. Hence, $\mathcal{R}(\cdot, f)$ is a contraction on $B(\rho_f)$ with a contraction constant

$$\begin{aligned}\frac{6\Delta(p)^3\beta}{\Theta}\rho_f^2 &= 4\sin^2\left(\frac{1}{3}\arcsin\left(|f|\sqrt{\frac{27\Delta(p)^3\beta}{2\Theta^3}}\right)\right)\\ &< 4\sin^2\left(\frac{1}{3}\arcsin 1\right) = 1.\end{aligned}$$

The proof of the uniqueness and existence is finished by the Banach fixed point theorem. Let $f_1, f_2 \in \mathbb{R}$ satisfy (14), then $U(f_i) \in B(\rho_{f_i}) \subset B(\rho_{f_3})$ for $i = 1, 2$ and $f_3 := \max\{|f_1|, |f_2|\}$. Note that f_3 satisfies (14). By (5), (6) and (13), we derive

$$\begin{aligned}\|U(f_1) - U(f_2)\|_1 &= \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_2)\|_1\\ &\le \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_1)\|_1 + \|\mathcal{R}(U(f_2), f_1) - \mathcal{R}(U(f_2), f_2)\|_1\\ &\le \frac{6\Delta(p)^3\beta}{\Theta}\rho_{f_3}^2\|U(f_1) - U(f_2)\|_1 + \frac{1}{\Theta}|f_1 - f_2|\\ &\le 4\sin^2\left(\frac{1}{3}\arcsin\left(f_3\sqrt{\frac{27\Delta(p)^3\beta}{2\Theta^3}}\right)\right)\|U(f_1) - U(f_2)\|_1\\ &\quad + \frac{1}{\Theta}|f_1 - f_2|,\end{aligned}$$

which implies (16). □

Theorem 2 *If*

$$|f| = \sqrt{\frac{2\Theta^3}{27\Delta(p)^3\beta}}, \tag{18}$$

then (2) *has a solution* $U(f) \in B(\rho_f)$ *where*

$$\rho_f = \frac{3|f|}{2\Theta}. \tag{19}$$

Proof We already know that $\mathcal{R}(\cdot, f)$ maps $B(\rho_f)$ into itself, and $\mathcal{R}(\cdot, f)$ is compact. The Schauder fixed point theorem implies the result. Note (15) and (18) give

$$\rho_f = \sqrt{\frac{\Theta}{6\Delta(p)^3\beta}} = \frac{3|f|}{2\Theta},$$

which is just (19).

Finally, we study (2) for $\gamma > 0$ under (3) by taking the homotopy (see [1])

$$\begin{aligned}\omega^2 U''(z) = \lambda\Big(&\alpha\,(U(z+p) + U(z-p) - 2U(z)) + \beta(U(z+p) - U(z))^3\\ &+\beta(U(z-p) - U(z))^3 - \gamma\omega U'(z) + f\cos z\Big), \quad \lambda \in [0,1].\end{aligned} \tag{20}$$

Putting

$$\begin{aligned}\mathcal{K}_1 U &:= \omega^2 U''(z),\\ \mathcal{F}_1(U, f) &:= \Big(\alpha\,(U(z+p) + U(z-p) - 2U(z))\\ &\quad +\beta(U(z+p) - U(z))^3 + \beta(U(z-p) - U(z))^3 - \gamma\omega U'(z) + f\cos z\Big),\end{aligned}$$

(20) has the form

$$\begin{aligned} U - \lambda\mathcal{R}_1(U, f) &= 0, \quad U \in Y, \quad \lambda \in [0,1],\\ \mathcal{R}_1(U, f) &:= \mathcal{K}_1^{-1}\mathcal{F}_1(U, f). \end{aligned} \tag{21}$$

Now, assume that (21) has a solution. Then (20) holds, and multiplying it by $U'(z)$ and integrating from 0 to 2π we obtain

$$\gamma\omega\|U'\|_{L^2}^2 = |f| \int_0^{2\pi} U'(z)\cos z dz \le |f|\|\cos z\|_{L^2}\|U'\|_{L^2},$$

which implies

$$\|U'\|_{L^2} \le \frac{|f|\sqrt{\pi}}{\gamma\omega}, \tag{22}$$

where $\|U\|_{L^2} := \sqrt{\int_0^{2\pi} U(z)^2 dz}$ is the usual norm on $L^2(0, 2\pi)$. Since $\sum_{k\in\mathbb{Z}} \frac{1}{(2k+1)^2} = \frac{\pi^2}{4}$ and $U(z) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{(2k+1)\imath z}$, by (22), we have

$$\begin{aligned}\|U\|_0 &:= \max_{z\in[0,2\pi]} |U(z)| \le \sum_{k\in\mathbb{Z}} |c_k|\\ &\le \sqrt{\sum_{k\in\mathbb{Z}} \frac{1}{(2k+1)^2}}\sqrt{\sum_{k\in\mathbb{Z}} (2k+1)^2 c_k^2} \le \frac{\sqrt{\pi}}{2\sqrt{2}}\|U'\|_{L^2} \le \frac{|f|\pi}{2\sqrt{2}\gamma\omega},\end{aligned} \tag{23}$$

which by (20), (22) and $\|U\|_{L^2} \leq \sqrt{2\pi}\|U\|_0$ gives

$$\omega^2\|U''\|_{L^2} \leq 4\alpha\sqrt{2\pi}\|U\|_0 + 16\beta(\sqrt{2\pi}\|U\|_0)^3 + \gamma\omega\|U'\|_{L^2} + |f|\sqrt{\pi}$$
$$\leq \frac{2|f|\sqrt{\pi}\left(\alpha\gamma^2\omega^2\pi + \beta f^2\pi^4 + \gamma^3\omega^3\right)}{\gamma^3\omega^3},$$

and so

$$\|U\|_1 = \sum_{k\in\mathbb{Z}}|2k+1||c_k| \leq \sqrt{\sum_{k\in\mathbb{Z}}\frac{1}{(2k+1)^2}}\sqrt{\sum_{k\in\mathbb{Z}}(2k+1)^4c_k^2}$$
$$\leq \frac{\sqrt{\pi}}{2\sqrt{2}}\|U''\|_{L^2} \leq \frac{|f|\pi\left(\alpha\gamma^2\omega^2\pi + \beta f^2\pi^4 + \gamma^3\omega^3\right)}{\sqrt{2}\gamma^3\omega^3}. \tag{24}$$

Summarizing, any solution of (21) belongs to the interior of $B(\widetilde{\rho}_f)$ for

$$\widetilde{\rho}_f := \frac{\sqrt{2}|f|\pi\left(\alpha\gamma^2\omega^2\pi + \beta f^2\pi^4 + \gamma^3\omega^3\right)}{\gamma^3\omega^3}.$$

This gives that the only solution of (2) for $\gamma > 0$ and $f = 0$ under (3) is the zero one $U = 0$. But since we suppose $f \neq 0$, the Leray-Schauder degree can be computed as follows

$$\deg\left(U - \mathcal{R}_1(U, f), B(\widetilde{\rho}_f), 0\right) = \deg\left(\mathbb{I}, B(\widetilde{\rho}_f), 0\right) = 1.$$

Consequently, we obtain the following result.

Theorem 3 *If $\gamma > 0$ then (2) under (3) has a solution in $U \in B(\widetilde{\rho}_f)$. Moreover, any solution of (2) under (3) belongs to the interior of $B(\widetilde{\rho}_f)$.*

For $\gamma = 0$ similar result could be derived like in Theorem 3 by using variational methods [14,16,15,32], but we do not present them since they are out of the scope of this paper.

Remark 1
1. If $\gamma > 0$ then (12) holds and $\Theta \geq \gamma\omega$, so we can replace Θ with $\gamma\omega$ in the above considerations.
2. If $\omega > 2\sqrt{\alpha}$ then (12) holds and $\Theta \geq \sqrt{\left(\omega^2 - 4\alpha\right)^2 + \gamma^2\omega^2}$, so now we can replace Θ with $\sqrt{\left(\omega^2 - 4\alpha\right)^2 + \gamma^2\omega^2}$ in the above considerations.
3. If $\gamma = 0$ then (12) is equivalent to

$$\omega \notin \left\{2\sqrt{\alpha}\left|\frac{\sin\frac{2k+1}{2}p}{2k+1}\right|\right\}_{k\in\mathbb{Z}}.$$

   Note that $\lim_{|k|\to\infty}\frac{\sin\frac{2k+1}{2}p}{2k+1} = 0$. So for any fixed $\omega > 0$, there is at most a finite number of resonant modes $k_0 \in \mathbb{Z}$ determined by equation

$$\left|\frac{\sin\frac{2k_0+1}{2}p}{2k_0+1}\right| = \frac{\omega}{2\sqrt{\alpha}}, \tag{25}$$

   where bifurcations could be studied of small solutions $U \in Y$ of (2) for f small.
4. By (8), we can replace $\Delta(p)$ with $|p|$ in (14). Consequently, (14) holds either if $\gamma > 0$ and

$$|f| < \sqrt{\frac{2\gamma^3\omega^3}{27|p|^3\beta}},$$

or if $\omega > 2\sqrt{\alpha}$ and

$$|f| < \sqrt{\frac{2}{27|p|^3\beta}}\left(\left(\omega^2 - 4\alpha\right)^2 + \gamma^2\omega^2\right)^{3/4}.$$

5. When we take $p = -\pi$ in (2), then using (3), we obtain

$$\omega^2 U''(z) + 4\alpha U(z) + 16\beta U(z)^3 + \gamma\omega U'(z) = f\cos z \tag{26}$$

which is just a forced and damped Duffing equation. Theorem 3 is applicable for (26). Next, suppose that we have 2 solutions U_1 and U_2 of (26) with (3). Then $V = U_1 - U_2$ satisfies

$$\omega^2 V''(z) + \left(4\alpha + 16\beta\left(U_1(z)^2 + U_2(z)U_2(z) + U_2(z)^2\right)\right)V(z) + \gamma\omega V'(z) = 0. \tag{27}$$

Since $U_{1,2}$ satisfy (23), we obtain

$$0 \le \left(U_1(z) + \frac{U_2(z)}{2}\right)^2 + \frac{3U_2(z)^2}{4} = U_1(z)^2 + U_2(z)U_2(z) + U_2(z)^2 \le \frac{3f^2\pi^2}{8\gamma^2\omega^2}.$$

Using this it follows that if

$$3\beta f^2\pi^2 < \gamma^3\sqrt{\alpha}\omega^2, \tag{28}$$

then condition (3.10) of [21, Lemma 3.7] holds and so (27) has the only solution $V = 0$, i.e., under (28) Eq. (26) has a unique solution with (3). Instead of (28), using [21, Lemma 3.2], we can use

$$f^2 \le \frac{\gamma^3\omega^2\left(\gamma + \sqrt{64\alpha + \gamma^2}\right)}{24\beta\pi^2}. \tag{29}$$

Inequality (28) or (29) provides a global uniqueness result for solutions satisfying the condition (3).

Scaling the parameters $\gamma \to \varepsilon\gamma$ and $f \to \varepsilon f$ in (26), with $\varepsilon \neq 0$, we can also apply the Melnikov subharmonic bifurcation theory [10]. Indeed, for $\varepsilon = 0$ we have

$$\omega^2 U''(z) + 4\alpha U(z) + 16\beta U(z)^3 = 0 \tag{30}$$

possessing a family of periodic solutions

$$U_a(z) = \frac{a}{2}\sqrt{\frac{\alpha}{\beta}}\,\mathrm{cn}\left(\sqrt{1+a^2}\,\frac{2\sqrt{\alpha}}{\omega}z\right)$$

for $a > 0$ with periods $T(a) = \frac{2K(k)\omega}{\sqrt{\alpha(1+a^2)}}$, $k = \frac{a}{\sqrt{2+2a^2}}$. Here, cn is the Jacobi elliptic function, $K(k)$ is the complete elliptic function of the first kind and k is the elliptic modulus [28]. Note that $U_a(0) = \frac{a}{2}\sqrt{\frac{\alpha}{\beta}}$, $U_a'(0) = 0$ and $U_a\left(z + \frac{2n+1}{2}T(a)\right) = -U_a(z)$ for any $n \in \mathbb{N}$. Moreover, $T(a)$ is decreasing from $T(0) = \frac{\pi\omega}{\sqrt{\alpha}}$ to 0. Then there is a sequence $\{a_n\}$ solving $T(a_n) = \frac{2\pi}{2n+1}$ for $n \in \mathbb{N}$, $2n+1 > \frac{2\sqrt{\alpha}}{\omega}$. Hence $\{U_{a_n}(z)\}$ is an unbounded sequence of 2π periodic solutions of (30) which also satisfy (3). Then using the above mentioned Melnikov method we could find ranges of γ and f for which many of $U_{a_n}(z)$ may persist after perturbation (see [13] for more details).

6. We would like to comment on the results and methods of [25,26]. All equations mentioned in this paragraph refer to those in the two references. The forms (2) of [25,26] and (17) of [26] of solutions of (1) consist of travelling waves $e^{i(\omega t \pm pn)}$ and $a_p(t)$. Then starting from (2), $\ddot{a}_p$ is neglected to obtain (3), an approach that is called as a rotating wave approximation. Equation (3) is an infinite system of ODEs, i.e., infinite ODE (1) is transformed into infinite ODE (3). The next argument is that all except a_k tend to zero as $t \to \infty$. Hence, the asymptotic property of (3) is studied. Therefore, it is claimed in [25,26] that the ω-limit set of (3) is (7), which is not necessarily the case. Hence, Eq. (7) is a first-order approximation of travelling waves when only two modes are considered and when the second derivative in (1) is neglected. This reminds us to the harmonic balance method [6]. Another multi-mode approximation is considered in (19). As for Eq. (17) in [26], it is clearly not a travelling wave solution. Our work complements those studies.

Khomeriki et al. [25,26] also discussed critical drives f_-, f_+, f_{cr}^{int} for the existence and stability of travelling waves. There are three travelling waves for $f_+ < f < f_-$ and the natural mode becomes unstable at $f_+ < f = f_{cr}^{int} < f_-$ [25]. When $k = \pi$, it was obtained that [25,26]

$$f^2 = 144z^3 + 24\left(4 - \omega^2\right)z^2 + \left[\left(4 - \omega^2\right)^2 + \gamma^2\omega^2\right]z, \tag{31}$$

where z is the amplitude squared of the π-mode. The corresponding equation for the general value of k was given in [26]. The critical drives $f_\mp$ are the square-roots of the right-hand side of (31) with

$$z = -\frac{1}{36}\left(2\left(4 - \omega^2\right) \pm \sqrt{\left(4 - \omega^2\right)^2 - 3\gamma^2\omega^2}\right). \tag{32}$$

It will be discussed later in Sect. 5 that we observe $f_\mp$, but not f_{cr}^{int}.

3 1D Forced FPU Lattices with Nonlocal Interactions

The existence and uniqueness analysis above can also be extended to the case of nonlocal couplings between the lattice sites. Here, we consider a general 1D damped FPU lattice forced by a travelling wave field with nonlocal interactions given by

$$\ddot{u}_n = \sum_{j\in\mathbb{Z}} \alpha_j\left(u_{n+j} - u_n\right) + \sum_{j\in\mathbb{Z}} \beta_j (u_{n+j} - u_n)^3 - \gamma\dot{u}_n + f\cos(\omega t + pn), \tag{33}$$

where $\alpha_j = \alpha_{-j} > 0$, $\alpha_0 = 0$, $\beta_j = \beta_{-j} > 0$, $\beta_0 = 0$ for any $j \in \mathbb{Z}$ and $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters. Moreover, we suppose

$$\sum_{j\in\mathbb{N}} \alpha_j < \infty, \quad \sum_{j\in\mathbb{N}} \beta_j < \infty.$$

Then (33) has the form

$$\ddot{u}_n = \sum_{j\in\mathbb{N}} \alpha_j\left(u_{n+j} + u_{n-j} - 2u_n\right) + \sum_{j\in\mathbb{N}} \beta_j\left((u_{n+j} - u_n)^3 + (u_{n-j} - u_n)^3\right)$$
$$-\gamma\dot{u}_n + f\cos(\omega t + pn). \tag{34}$$

Putting $u_n(t) = U(\omega t + pn)$ for $U \in Y$ in (34), we obtain

$$\omega^2 U''(z) = \sum_{j\in\mathbb{N}} \alpha_j\left(U(z + pj) + U(z - pj) - 2U(z)\right) - \gamma\omega U'(z)$$

$$+\sum_{j\in\mathbb{N}}\beta_j\left((U(z+pj)-U(z))^3+(U(z-pj)-U(z))^3\right)+f\cos z. \tag{35}$$

By setting

$$\mathcal{J}U:=\omega^2U''(z)+\gamma\omega U'(z)-\sum_{j\in\mathbb{N}}\alpha_j\left(U(z+pj)+U(z-pj)-2U(z)\right),$$
$$\mathcal{G}(U,f):=\sum_{j\in\mathbb{N}}\beta_j\left((U(z+pj)-U(z))^3+(U(z-pj)-U(z))^3\right)+f\cos z,$$

(35) has the form

$$\mathcal{J}U=\mathcal{G}(U,f).$$

Following the proof of Lemma 2, we have the next result.

Lemma 3 *$\mathcal{G}:Y\times\mathbb{R}\to X$ satisfies*

$$\|\mathcal{G}(U,f)\|\le 2\sum_{j\in\mathbb{N}}\beta_j\Delta(pj)^3\|U\|_1^3+|f|,$$
$$\|\mathcal{G}(U_1,f)-\mathcal{G}(U_2,f)\|\le 2\sum_{j\in\mathbb{N}}\beta_j\Delta(pj)^3\|U_1-U_2\|_1\left(\|U_1\|_1^2+\|U_1\|_1\|U_2\|_1+\|U_2\|_1^2\right),$$
$$\|\mathcal{G}(U,f_1)-\mathcal{G}(U,f_2)\|\le|f_1-f_2|$$

for any $U,U_1,U_2\in Y$ and $f,f_1,f_2\in\mathbb{R}$.

Next if $U\in Z$ with $U(z)=\sum_{k\in\mathbb{Z}}c_k\mathrm{e}^{(2k+1)\imath z}$ then

$$\mathcal{J}U(z)=\sum_{k\in\mathbb{Z}}\left(4\sum_{j\in\mathbb{N}}\alpha_j\sin^2\frac{2k+1}{2}pj-(2k+1)^2\omega^2+\gamma\omega(2k+1)\imath\right)c_k\mathrm{e}^{(2k+1)\imath z}.$$

Hence $\mathcal{J}\in L(Z,X)$ with

$$\|\mathcal{J}\|_{L(Z,X)}\le 4\sum_{j\in\mathbb{N}}\alpha_j+\omega^2+\gamma\omega.$$

Moreover, assuming

$$4\sum_{j\in\mathbb{N}}\alpha_j\frac{\sin^2\frac{2k+1}{2}pj}{(2k+1)^2}-\omega^2+\frac{\gamma\omega}{2k+1}\imath\ne 0\quad\forall k\in\mathbb{Z}, \tag{36}$$

then

$$\Gamma:=\min_{k\in\mathbb{Z}}\sqrt{\left(4\sum_{j\in\mathbb{N}}\alpha_j\frac{\sin^2\frac{2k+1}{2}pj}{2k+1}-(2k+1)\omega^2\right)^2+\gamma^2\omega^2}>0.$$

So $\mathcal{J}^{-1}:X\to Y$ is compact such that

$$\|\mathcal{J}^{-1}\|_{L(X,Y)}\le\frac{1}{\Gamma}.$$

Summarizing, we arrive at the following result.

Theorem 4 *Suppose* (36)*. By replacing*

$$\beta\Delta(p)^3 \leftrightarrow \sum_{j\in\mathbb{N}} \beta_j \Delta(pj)^3, \quad \Theta \leftrightarrow \Gamma,$$

the statements of Theorems 1 and 2 are valid for (35)*.*

Theorem 3 and Remark 1 can be similarly modified.

4 Simple resonances

Here we study the case of simple resonances (see Remark 1.3), i.e. we suppose the assumption

(R1) For some $k_0 \in \mathbb{Z}$ Eq. (25) holds and has the only solutions $-1-k_0, k_0$.

We investigate the bifurcation of a small solution. So we consider the equation

$$A(U,\varepsilon) := \mathcal{K}U - \mathcal{F}(U) - \varepsilon\mathcal{G}(U) = 0, \tag{37}$$

where

$$\begin{aligned} \mathcal{K}U &= \omega^2 U''(z) - \alpha U(z+p) - \alpha U(z-p) + 2\alpha U(z), \\ \mathcal{F}(U) &= \beta(U(z+p) - U(z))^3 + \beta(U(z-p) - U(z))^3, \\ \mathcal{G}(U)(z) &= f\cos z - \gamma\omega U'(z) \end{aligned} \tag{38}$$

for U and ε small. For small $\varepsilon > 0$ and $\gamma \geq 0$, we are looking for f such that (37) is solvable. In further work we denote $\kappa := |2k_0 + 1|$.

We shall solve Eq. (37) via Lyapunov–Schmidt reduction method [10]. It holds $A(0,0) = 0, D_U A(0,0) = \mathcal{K}$. Let $R_1^\kappa = \mathcal{N}\mathcal{K}$, $R_2^\kappa = \mathcal{R}\mathcal{K}$ be the null space and the image of operator $\mathcal{K} \in L(Z,X)$, respectively. Then recalling (11) with $\gamma = 0$, if $U \in Z$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k e^{(2k+1)\imath z}$ then

$$\mathcal{K}U(z) = \sum_{k\in\mathbb{Z}\setminus\{-1-k_0,k_0\}} \left(4\alpha\sin^2\frac{2k+1}{2}p - (2k+1)^2\omega^2\right) c_k e^{(2k+1)\imath z}.$$

Hence

$$\begin{aligned} R_1^\kappa &= \left\{ c e^{\kappa\imath z} + \bar{c} e^{-\kappa\imath z},\ c \in \mathbb{C} \right\}, \\ R_2^\kappa &= \left\{ U \in X \mid U(z) = \sum_{k\in\mathbb{Z}\setminus\{-1-k_0,k_0\}} c_k e^{(2k+1)\imath z} \right\}. \end{aligned}$$

Let $Q : X \to R_2^\kappa$ be the projection onto $R_2^\kappa \subset X$ given by

$$QU(z) = \sum_{k\in\mathbb{Z}\setminus\{-1-k_0,k_0\}} c_k e^{(2k+1)\imath z}.$$

Now we take the decomposition $U = U_1 + U_2$, $U_1 \in R_1^\kappa$, $U_2 \in R_2^\kappa \cap Z$ for $U \in Z$, and decouple (37) to equations

$$QA(U_1 + U_2, \varepsilon) = 0, \tag{39}$$

$$(\mathbb{I} - Q)A(U_1 + U_2, \varepsilon) = 0. \tag{40}$$

From the first one, using the implicit function theorem we obtain the existence of neighbourhoods $V_1 \times V_0 \subset R_1^\kappa \times \mathbb{R}$ of $(0,0)$ in $R_1^\kappa \times \mathbb{R}$ and $V_2 \subset R_2^\kappa \cap Z$ of 0 in $R_2^\kappa \cap Z$ and a unique C^∞-function $U_2 : V_1 \times V_0 \to V_2$ such that $QA(U_1 + \widetilde{U}_2, \varepsilon) = 0$ for $(U_1, \varepsilon) \in V_1 \times V_0$, $\widetilde{U}_2 \in V_2$ if and only if $\widetilde{U}_2 = U_2(U_1, \varepsilon)$. Moreover $U_2(0,0) = 0$. In the next steps, we consider three different cases: $\kappa = 1$, $\kappa = 3$ and $\kappa \geq 5$.

Let $\kappa = 1$. By differentiating Eq. (39) we obtain the following derivatives of function U_2

$$\begin{aligned}
\mathrm{D}_{U_1} U_2(0,0) &= 0, \qquad \mathrm{D}_\varepsilon U_2(0,0) = -\mathcal{K}^{-1} Q \mathrm{D}_\varepsilon A(0,0) = 0,\\
\mathrm{D}_{U_1 U_1} U_2(0,0) &= -\mathcal{K}^{-1} Q \mathrm{D}_{UU}\ A(0,0)|_{R_1^1} = 0,\\
\mathrm{D}_{U_1 \varepsilon} U_2(0,0) &= -\mathcal{K}^{-1} Q \mathrm{D}_{U\varepsilon}\ A(0,0)|_{R_1^1} = \mathcal{K}^{-1} Q \mathrm{D}_U\ \mathcal{G}(0)|_{R_1^1} = 0,\\
\mathrm{D}_{\varepsilon\varepsilon} U_2(0,0) &= -2\mathcal{K}^{-1} Q \mathrm{D}_{U\varepsilon} A(0,0) \mathrm{D}_\varepsilon U_2(0,0) = 0,
\end{aligned}$$

where $|_{R_1^1}$ denotes the restriction on R_1^1. Thus $U_2(U_1, \varepsilon) = O(\|(U_1, \varepsilon)\|^3)$. Equation (40) now has the form

$$(\mathbb{I} - Q) A(U_1 + U_2(U_1, \varepsilon), \varepsilon) = 0$$

for $U_1 \in V_1$ and $\varepsilon \in V_0$. Expanding into Taylor series yields

$$\begin{aligned}
\mathcal{F}(U_1 + U_2(U_1, \varepsilon)) &= \frac{1}{3!} \mathrm{D}^3 \mathcal{F}(0) U_1^3 + O(\|(U_1, \varepsilon)\|^4),\\
\mathcal{G}(U_1 + U_2(U_1, \varepsilon)) &= \mathcal{G}(0) + \mathrm{D}\mathcal{G}(0) U_1 + O(\|(U_1, \varepsilon)\|^3).
\end{aligned}$$

Hence we obtain equation

$$(\mathbb{I} - Q) \left[\frac{1}{3!} \mathrm{D}^3 \mathcal{F}(0) U_1^3 + \varepsilon \mathcal{G}(0) + \varepsilon \mathrm{D}\mathcal{G}(0) U_1 + O(\|(U_1, \varepsilon)\|^4) \right] = 0.$$

We put $\mathcal{F}$ and $\mathcal{G}$ from (38)

$$\begin{aligned}
(\mathbb{I} - Q) \big[&\beta(U_1(z+p) - U_1(z))^3 + \beta(U_1(z-p) - U_1(z))^3\\
&+ \varepsilon f \cos z - \varepsilon\gamma\omega U_1'(z) + O(\|(U_1, \varepsilon)\|^4) \big] = 0.
\end{aligned} \tag{41}$$

For $U_1 = c\mathrm{e}^{\imath z} + \bar{c}\mathrm{e}^{-\imath z} \in R_1^1$ we obtain

$$U_1(z \pm p) - U_1(z) = c(\mathrm{e}^{\pm\imath p} - 1)\mathrm{e}^{\imath z} + \bar{c}(\mathrm{e}^{\mp\imath p} - 1)\mathrm{e}^{-\imath z}.$$

If we denote $d_\pm := c(\mathrm{e}^{\pm\imath p} - 1)$, then (41) has the form

$$\begin{aligned}
0 &= (\mathbb{I} - Q) \big[\beta(d_+ \mathrm{e}^{\imath z} + \overline{d_+} \mathrm{e}^{-\imath z})^3 + \beta(d_- \mathrm{e}^{\imath z} + \overline{d_-} \mathrm{e}^{-\imath z})^3 + \varepsilon f \cos z - \varepsilon\gamma\omega U_1'(z)\\
&\quad + O(\|(U_1, \varepsilon)\|^4) \big]\\
&= 3\beta(d_+^2 \overline{d_+} \mathrm{e}^{\imath z} + d_+ \overline{d_+}^2 \mathrm{e}^{-\imath z}) + 3\beta(d_-^2 \overline{d_-} \mathrm{e}^{\imath z} + d_- \overline{d_-}^2 \mathrm{e}^{-\imath z})\\
&\quad + \varepsilon f \cos z - \varepsilon\gamma\omega U_1'(z) + O(\|(U_1, \varepsilon)\|^4).
\end{aligned}$$

By using exponential form of $\cos z$ and collecting coefficients we derive

$$\begin{aligned}
&\left[3\beta(d_+^2 \overline{d_+} + d_-^2 \overline{d_-}) + \varepsilon f/2 - \varepsilon\gamma\omega\imath c \right] \mathrm{e}^{\imath z} + \overline{\left[3\beta(d_+^2 \overline{d_+} + d_-^2 \overline{d_-}) + \varepsilon f/2 - \varepsilon\gamma\omega\imath c \right]} \mathrm{e}^{-\imath z}\\
&\quad + O(\|(U_1, \varepsilon)\|^4) = 0.
\end{aligned}$$

This is equivalent to

$$3\beta(d_+^2 \overline{d_+} + d_-^2 \overline{d_-}) + \varepsilon f/2 - \varepsilon\gamma\omega\imath c + O(\|(c, \varepsilon)\|^4) = 0$$

or when one returns to c

$$-48\beta c|c|^2 \sin^4 \frac{p}{2} + \varepsilon \frac{f}{2} - \varepsilon\gamma\omega\imath c + O(\|(c,\varepsilon)\|^4) = 0.$$

Considering real and imaginary parts yields

$$\begin{aligned} -48\beta x(x^2+y^2)\sin^4 \frac{p}{2} + \varepsilon\frac{f}{2} + \varepsilon\gamma\omega y + O(\|(x,y,\varepsilon)\|^4) &= 0 \\ -48\beta y(x^2+y^2)\sin^4 \frac{p}{2} - \varepsilon\gamma\omega x + O(\|(x,y,\varepsilon)\|^4) &= 0 \end{aligned}$$

for $c = x + \imath y \in \mathbb{C}$, $x, y \in \mathbb{R}$. Now we put the resonance condition $\left|\sin \frac{p}{2}\right| = \frac{\omega}{2\sqrt{\alpha}}$ to get rid of parameter p and obtain system

$$\begin{aligned} a\varepsilon y + b\varepsilon - \tilde{c}x(x^2+y^2) + O(\|(x,y,\varepsilon)\|^4) &= 0 \\ -a\varepsilon x - \tilde{c}y(x^2+y^2) + O(\|(x,y,\varepsilon)\|^4) &= 0, \end{aligned} \tag{42}$$

where

$$a = \gamma\omega \geq 0, \qquad b = f/2 \neq 0, \qquad \tilde{c} = 3\beta\omega^4/\alpha^2 > 0. \tag{43}$$

We scale

$$x \longleftrightarrow \varepsilon x, \qquad y \longleftrightarrow \varepsilon y, \qquad f \longleftrightarrow \varepsilon^2 f, \qquad \gamma \longleftrightarrow \varepsilon\gamma \tag{44}$$

and divide both equations by ε^3 to obtain the bifurcation equation for Eq. (37)

$$\begin{aligned} ay + b - \tilde{c}x(x^2+y^2) + \varepsilon O(\|(x,y,1)\|^4) &= 0 \\ -ax - \tilde{c}y(x^2+y^2) + \varepsilon O(\|(x,y,1)\|^4) &= 0. \end{aligned} \tag{45}$$

This equation will be solved later.

Let $\kappa = 3$. Then differentiating Eq. (39) yields

$$\begin{aligned} \mathrm{D}_{U_1}U_2(0,0) = 0, \qquad \mathrm{D}_\varepsilon U_2(0,0) &= -\mathcal{K}^{-1}Q\mathrm{D}_\varepsilon A(0,0) = \mathcal{K}^{-1}\mathcal{G}(0), \\ \mathrm{D}_{U_1U_1}U_2(0,0) &= -\mathcal{K}^{-1}Q\mathrm{D}_{UU}\ A(0,0)|_{R_1^3} = 0, \\ \mathrm{D}_{U_1\varepsilon}U_2(0,0) &= -\mathcal{K}^{-1}Q\mathrm{D}_{U\varepsilon}\ A(0,0)|_{R_1^3} = \mathcal{K}^{-1}Q\mathrm{D}_U\ \mathcal{G}(0)|_{R_1^3} = 0, \\ \mathrm{D}_{\varepsilon\varepsilon}U_2(0,0) &= -2\mathcal{K}^{-1}Q\mathrm{D}_{U\varepsilon}A(0,0)\mathrm{D}_\varepsilon U_2(0,0) = 2\mathcal{K}^{-1}\mathrm{D}\mathcal{G}(0)(\mathcal{K}^{-1}\mathcal{G}(0)), \end{aligned} \tag{46}$$

where we made the restrictions on R_1^3. Hence we obtain

$$U_2(U_1,\varepsilon) = \mathcal{K}^{-1}\mathcal{G}(0)\varepsilon + \mathcal{K}^{-1}\mathrm{D}\mathcal{G}(0)(\mathcal{K}^{-1}\mathcal{G}(0))\varepsilon^2 + O(\|(U_1,\varepsilon)\|^3).$$

This time from Taylor series

$$\begin{aligned} \mathcal{F}(U_1 + U_2(U_1,\varepsilon)) &= \frac{1}{3!}\mathrm{D}^3\mathcal{F}(0)(U_1 + \mathrm{D}_\varepsilon U_2(0,0)\varepsilon)^3 + O(\|(U_1,\varepsilon)\|^4), \\ \mathcal{G}(U_1 + U_2(U_1,\varepsilon)) &= \mathcal{G}(0) + \mathrm{D}\mathcal{G}(0)\left(U_1 + \mathrm{D}_\varepsilon U_2(0,0)\varepsilon + \frac{1}{2}\mathrm{D}_{\varepsilon\varepsilon}U_2(0,0)\varepsilon^2\right) \\ &\quad + O(\|(U_1,\varepsilon)\|^3). \end{aligned}$$

We put $\mathcal{F}$ and $\mathcal{G}$ from (38) into these expansions and, next, into Eq. (40). So we obtain

$$\begin{aligned} (\mathbb{I} - Q)\big[&\beta(U_1(z+p) - U_1(z) + \varepsilon f\mathcal{K}^{-1}(\cos(z+p) - \cos z))^3 \\ &+\beta(U_1(z-p) - U_1(z) + \varepsilon f\mathcal{K}^{-1}(\cos(z-p) - \cos z))^3 + \varepsilon f\cos z \end{aligned}$$

$$-\varepsilon\gamma\omega(U_1'(z) + (\mathcal{K}^{-1}\cos z)' f\varepsilon - (\mathcal{K}^{-1}(\mathcal{K}^{-1}\cos z)')' f\gamma\omega\varepsilon^2)$$
$$+ O(\|(U_1,\varepsilon)\|^4)\big] = 0. \tag{47}$$

Since

$$U_1(z \pm p) - U_1(z) = d_\pm e^{3\imath z} + \overline{d_\pm} e^{-3\imath z}$$

for $U_1(z) = ce^{3\imath z} + \bar{c}e^{-3\imath z} \in R_1^3$, $d_\pm = c(e^{\pm 3\imath p} - 1)$ and

$$\cos(z \pm p) - \cos z = \frac{(e^{\pm\imath p} - 1)e^{\imath z} + (e^{\mp\imath p} - 1)e^{-\imath z}}{2},$$

after applying the projection $\mathbb{I} - Q$ we obtain from (47)

$$Ce^{3\imath z} + \overline{C}e^{-3\imath z} + O(\|(U_1,\varepsilon)\|^4) = 0 \tag{48}$$

with constant C given by

$$C := \beta\left(3(d_+ + d_-)|d_+|^2 + \frac{3\varepsilon^2 f^2 k^2}{2}(d_+ + d_-)|e^{\imath p} - 1|^2 + \frac{\varepsilon^3 f^3 k^3}{8}((e^{\imath p} - 1)^3 + (e^{-\imath p} - 1)^3)\right) - 3\varepsilon\gamma\omega\imath c$$

and

$$k = \frac{1}{4\alpha\sin^2\frac{p}{2} - \omega^2}.$$

Note that due to assumption (R1), k is well-defined. Clearly, Eq. (48) is satisfied if and only if $C + O(\|(U_1,\varepsilon)\|^4) = 0$, i.e. when one returns to c

$$\beta\left(-48c|c|^2\sin^4\frac{3}{2}p - 24\varepsilon^2 f^2k^2 c\sin^2\frac{p}{2}\sin^2\frac{3}{2}p + 2\varepsilon^3 f^3k^3\sin^3\frac{p}{2}\sin\frac{3}{2}p\right)$$
$$-3\varepsilon\gamma\omega\imath c + O(\|(c,\bar{c},\varepsilon)\|^4) = 0.$$

Now we separate real and imaginary part, apply scaling (44) and divide by ε^3 to obtain bifurcation equation for Eq. (37) and $\kappa = 3$

$$ay + b - dx - \tilde{c}x(x^2 + y^2) + \varepsilon O(\|(x, y, 1)\|^4) = 0$$
$$-ax - dy - \tilde{c}y(x^2 + y^2) + \varepsilon O(\|(x, y, 1)\|^4) = 0, \tag{49}$$

where $c = x + \imath y$ and

$$a = 3\gamma\omega \geq 0, \qquad b = 2\beta f^3k^3\sin^3\frac{p}{2}\sin\frac{3}{2}p \neq 0,$$
$$\tilde{c} = 48\beta\sin^4\frac{3}{2}p > 0, \qquad d = 24\beta f^2k^2\sin^2\frac{p}{2}\sin^2\frac{3}{2}p > 0. \tag{50}$$

Once again, solving this equation is left for the below-stated Theorem 6.

Let $\kappa \geq 5$. Then we have the same derivatives for U_2 as in (46) but with restrictions on R_1^κ, i.e.

$$U_2(U_1,\varepsilon) = \mathcal{K}^{-1}\mathcal{G}(0)\varepsilon + \mathcal{K}^{-1}D\mathcal{G}(0)(\mathcal{K}^{-1}\mathcal{G}(0))\varepsilon^2 + O(\|(U_1,\varepsilon)\|^3).$$

Exactly as for $\kappa = 3$ we derive Eq. (47). Now for $U_1(z) = ce^{\kappa\imath z} + \bar{c}e^{-\kappa\imath z} \in R_1^\kappa$ it holds

$$U_1(z \pm p) - U_1(z) = d_\pm e^{\kappa\imath z} + \overline{d_\pm}e^{-\kappa\imath z}$$

with $d_\pm = c(\mathrm{e}^{\pm\kappa\imath z} - 1)$. Therefore, from (47) applying projection $\mathbb{I} - Q$ we obtain

$$C\mathrm{e}^{\kappa\imath z} + \overline{C}\mathrm{e}^{-\kappa\imath z} + O(\|(U_1,\varepsilon)\|^4) = 0,$$

where

$$\begin{aligned} C &:= \beta\left(3(d_+ + d_-)|d_+|^2 + \frac{3\varepsilon^2 f^2 k^2}{2}(d_+ + d_-)|\mathrm{e}^{\imath p} - 1|^2\right) - \kappa\varepsilon\gamma\omega\imath c \\ &= -48\beta c|c|^2 \sin^4\frac{\kappa}{2}p - 24\beta\varepsilon^2 f^2 k^2 c \sin^2\frac{p}{2}\sin^2\frac{\kappa}{2}p - \kappa\varepsilon\gamma\omega\imath c. \end{aligned}$$

The last equation is equivalent to $C + O(\|(c,\bar{c},\varepsilon)\|^4) = 0$. Then separating real and imaginary part, scaling (44) and by dividing both equations by ε^3 we arrive at the bifurcation equation for Eq. (37) and $\kappa \geq 5$ given by

$$\begin{aligned} &ay - dx - \tilde{c}x(x^2+y^2) + \varepsilon O(\|(x,y,1)\|^4) = 0 \\ &-ax - dy - \tilde{c}y(x^2+y^2) + \varepsilon O(\|(x,y,1)\|^4) = 0, \end{aligned} \tag{51}$$

where $c = x + \imath y$ and

$$a = \kappa\gamma\omega \geq 0, \quad \tilde{c} = 48\beta\sin^4\frac{\kappa}{2}p > 0, \quad d = 24\beta f^2 k^2 \sin^2\frac{p}{2}\sin^2\frac{\kappa}{2}p > 0. \tag{52}$$

In conclusion, using scaling (44) we derived bifurcation equations for Eq. (37) and any $\kappa \in \{|2k+1| \mid k \in \mathbb{Z}\}$. To prove our result, first we recall the Banach and Mazur theorem [2].

Theorem 5 *$F \in C(\mathbb{R}^n, \mathbb{R}^n)$ is a (global) homeomorphism if and only if it is local and coercive, i.e. $\|F(x)\| \to \infty$ as $\|x\| \to \infty$.*

Now we state the result on the existence of a small solution of (37).

Theorem 6 *Assume* (R1). *Then if $\kappa = 1$ and $\gamma > 0$, or $\kappa \geq 3$, Eq.* (37) *has a solution $U \in Z$ close to 0 for any $\varepsilon \neq 0$ sufficiently small.*

Proof Define

$$\widetilde{H}_1(x,y) = \big(ay - \tilde{c}x(x^2+y^2), -ax - \tilde{c}y(x^2+y^2)\big)$$

with $a, \tilde{c}$ of (43),

$$\widetilde{H}_2(x,y) = \big(ay - dx - \tilde{c}x(x^2+y^2), -ax - dy - \tilde{c}y(x^2+y^2)\big)$$

with $a, \tilde{c}, d$ of (50) and

$$\widetilde{H}_3(x,y) = \big(ay - dx - \tilde{c}x(x^2+y^2), -ax - dy - \tilde{c}y(x^2+y^2)\big)$$

with $a, \tilde{c}, d$ of (52). Then (45) has the form

$$H_1(x,y,q,\varepsilon) = \widetilde{H}_1(x,y) + (q,0) + \varepsilon O(\|(x,y,1)\|^4) = 0.$$

Since

$$\mathrm{D}_{(x,y)}\widetilde{H}_1(x,y) = \begin{pmatrix} -3\tilde{c}x^2 - \tilde{c}y^2 & a - 2\tilde{c}xy \\ -a - 2\tilde{c}xy & -\tilde{c}x^2 - 3\tilde{c}y^2 \end{pmatrix}$$

and $\det \mathrm{D}_{(x,y)}\widetilde{H}_1(x,y) = a^2 + 3\tilde{c}^2(x^2+y^2)^2 > 0$ for $\gamma > 0$, $\mathrm{D}_{(x,y)}\widetilde{H}_1(x,y)$ is regular for any $(x,y) \in \mathbb{R}^2$. Next it holds

$$\|\widetilde{H}_1(x,y)\| = \|(x,y)\|\sqrt{a^2 + \tilde{c}^2\|(x,y)\|^4} \to \infty$$

as $\|(x, y)\| = \sqrt{x^2 + y^2} \to \infty$. Hence Theorem 5 can be applied together with the implicit function theorem. Consequently, there exists a unique pair of continuous functions $x(q, \varepsilon)$, $y(q, \varepsilon)$ defined for ε close to 0 such that $H_1(\widetilde{x}, \widetilde{y}, q, \varepsilon) = 0$ if and only if $\widetilde{x} = x(q, \varepsilon)$ and $\widetilde{y} = y(q, \varepsilon)$. In addition, $(x(q, 0), y(q, 0))$ is the only solution of $\widetilde{H}_1(x, y) + (q, 0) = 0$. So we have the existence of a unique solution of the bifurcation Eq. (45) for any parameters $\alpha, \beta, \gamma, \omega > 0$, ε small and $f \neq 0$. After scaling backwards in (44) we obtain a solution of Eq. (37).

Analogically, (49) has the form

$$\widetilde{H}_2(x, y) + (q, 0) + \varepsilon O(\|(x, y, 1)\|^4) = 0,$$

and for derivative of $\widetilde{H}_2(x, y)$ we obtain

$$\mathrm{D}_{(x,y)} \widetilde{H}_2(x, y) = \begin{pmatrix} -d - 3\tilde{c}x^2 - \tilde{c}y^2 & a - 2\tilde{c}xy \\ -a - 2\tilde{c}xy & -d - \tilde{c}x^2 - 3\tilde{c}y^2 \end{pmatrix}$$

and $\det \mathrm{D}_{(x,y)} \widetilde{H}_2(x, y) = a^2 + d^2 + 4\tilde{c}d(x^2 + y^2) + 3\tilde{c}^2(x^2 + y^2)^2 > 0$ for any $\gamma \geq 0$. Also it holds

$$\|\widetilde{H}_2(x, y)\| = \|(x, y)\| \sqrt{a^2 + \left(d + \tilde{c}\|(x, y)\|^2\right)^2} \to \infty$$

as $\|(x, y)\| \to \infty$. So we obtain the existence of a unique solution of bifurcation Eq. (49) and scaling backward in (44) the existence of a solution of Eq. (37) for $\kappa = 3$.

Finally, (51) has the form

$$\widetilde{H}_3(x, y) + \varepsilon O(\|(x, y, 1)\|^4) = 0,$$

and observing that formally $\widetilde{H}_3(x, y) = \widetilde{H}_2(x, y)$, we can repeat the above arguments. Therefore, it is regular for any $\gamma \geq 0$ and it follows the existence of a unique solution of bifurcation Eq. (51) and the existence of a solution of Eq. (37) for $\kappa \geq 5$. Note now $\widetilde{x} = O(\varepsilon)$ and $\widetilde{y} = O(\varepsilon)$.

Remark 2 1. Using another scaling we can obtain another range of parameters ε, f and γ for which there are solutions of (37). To demonstrate this, we focus on the case $\kappa = 1$. First, we scale (42) by

$$x \longleftrightarrow \varepsilon x, \qquad y \longleftrightarrow \varepsilon y, \qquad f \longleftrightarrow \varepsilon^2 f, \qquad \gamma \longleftrightarrow \varepsilon^2 \gamma, \tag{53}$$

divide both equations by ε^3 and obtain

$$\begin{aligned} b - \tilde{c}x(x^2 + y^2) + a\varepsilon y + \varepsilon O(\|(x, y, 1)\|^4) &= 0 \\ -\tilde{c}y(x^2 + y^2) - a\varepsilon x + \varepsilon O(\|(x, y, 1)\|^4) &= 0. \end{aligned} \tag{54}$$

If we define

$$\begin{aligned} \bar{H}_1(x, y, \varepsilon) = (b - \tilde{c}x(x^2 + y^2) + a\varepsilon y + \varepsilon O(\|(x, y, 1)\|^4), \\ -\tilde{c}y(x^2 + y^2) - a\varepsilon x + \varepsilon O(\|(x, y, 1)\|^4)), \end{aligned}$$

where $a, b, \tilde{c}$ are given by (43), we obtain $\bar{H}_1\left(\sqrt[3]{\frac{b}{\tilde{c}}}, 0, 0\right) = 0$ and

$$\mathrm{D}_{(x,y)} \bar{H}_1\left(\sqrt[3]{\frac{b}{\tilde{c}}}, 0, 0\right) = \begin{pmatrix} -3\widetilde{c}\left(\frac{b}{\tilde{c}}\right)^{\frac{2}{3}} & 0 \\ 0 & -\tilde{c}\left(\frac{b}{\tilde{c}}\right)^{\frac{2}{3}} \end{pmatrix}.$$

Then $\det \mathrm{D}_{(x,y)}\bar{H}_1\left(\sqrt[3]{\frac{b}{\tilde{c}}}, 0, 0\right) = 3\tilde{c}^2\left(\frac{b}{\tilde{c}}\right)^{\frac{4}{3}} > 0$ and, consequently, there is a unique solution $(x, y) = (x(\varepsilon), y(\varepsilon))$ of Eq. (54) for ε sufficiently small such that $(x(0), y(0)) = \left(\sqrt[3]{\frac{b}{\tilde{c}}}, 0\right)$. Scaling backwards in (53) we obtain the existence of a solution of Eq. (37), which is other than the one proved in Theorem 6. This solution exists even for $\gamma = 0$.

Furthermore, another scaling

$$x \longleftrightarrow \varepsilon x, \qquad y \longleftrightarrow \varepsilon y, \qquad f \longleftrightarrow \varepsilon f, \qquad \gamma \longleftrightarrow \gamma, \tag{55}$$

of (42) gives

$$\begin{aligned} &ay + b + \varepsilon O(\|(x, y, 1)\|^4) = 0 \\ &-ax + \varepsilon O(\|(x, y, 1)\|^4) = 0. \end{aligned} \tag{56}$$

There is a unique solution $(x, y) = (\bar{x}(\varepsilon), \bar{y}(\varepsilon))$ of Eq. (56) for ε sufficiently small such that $(\bar{x}(0), \bar{y}(0)) = \left(0, -\frac{b}{a}\right)$. Scaling backwards in (55) we obtain the existence of a solution of Eq. (37), which is other than the above ones. So we see that for $\kappa = 1$ and $f > 0$, $\gamma > 0$ small, there are at least 3 different branches of solutions of (37). Of course, we need to scale back by using (44), (53) and (55) to obtain appropriate values of f and γ. The order of solutions is $O(\varepsilon)$ in Theorem 6.

Finally, considering for $m_1, n_1 \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $k_1, r_1, s_1 \in \mathbb{N}$ a general scaling

$$x \longleftrightarrow \varepsilon^{r_1} x, \qquad y \longleftrightarrow \varepsilon^{s_1} y, \qquad f \longleftrightarrow \varepsilon^{m_1} f, \qquad \gamma \longleftrightarrow \varepsilon^{n_1}\gamma, \qquad \varepsilon \longleftrightarrow \varepsilon^{k_1} \tag{57}$$

of (42) gives

$$\begin{aligned} &ay\varepsilon^{k_1+n_1+s_1} + b\varepsilon^{k_1+m_1} - \tilde{c}x\varepsilon^{r_1}(x^2\varepsilon^{2r_1} + y^2\varepsilon^{2s_1}) + O\left(\left(|\varepsilon|^{r_1} + |\varepsilon|^{s_1} + |\varepsilon|^{k_1}\right)^4\right) = 0 \\ &-ax\varepsilon^{k_1+n_1+r_1} - \tilde{c}y\varepsilon^{s_1}(x^2\varepsilon^{2r_1} + y^2\varepsilon^{2s_1}) + O\left(\left(|\varepsilon|^{r_1} + |\varepsilon|^{s_1} + |\varepsilon|^{k_1}\right)^4\right) = 0. \end{aligned} \tag{58}$$

Taking $r_1 = s_1 = 1$, $m_1 = n_1 = 0$ and $k_1 = 3$, we derive

$$\begin{aligned} &ay\varepsilon + b - \tilde{c}x(x^2 + y^2) + O(\varepsilon) = 0 \\ &-ax\varepsilon - \tilde{c}y(x^2 + y^2) + O(\varepsilon) = 0. \end{aligned} \tag{59}$$

Since $b \neq 0$, there is a unique solution $(x, y) = (\bar{x}(\varepsilon), \bar{y}(\varepsilon))$ of Eq. (59) for ε sufficiently small such that $(\bar{x}(0), \bar{y}(0)) = \left(\sqrt[3]{\frac{b}{\tilde{c}}}, 0\right)$ (see (54)). Note that this solution of (37) is of order $O(\sqrt[3]{\varepsilon})$ while f and γ are not scaled.

Taking $r_1 = s_1 = 1$, $m_1 = 1$, $n_1 = 0$ and $k_1 = 2$, we derive

$$\begin{aligned} &ay + b - \tilde{c}x(x^2 + y^2) + O(\varepsilon) = 0 \\ &-ax - \tilde{c}y(x^2 + y^2) + O(\varepsilon) = 0. \end{aligned} \tag{60}$$

We obtain an equation similar to (45). So there is a unique solution of Eq. (60) for ε sufficiently small and $\gamma > 0$. Note that this solution of (37) is of order $O(\sqrt{\varepsilon})$, $\varepsilon > 0$ while $f \leftrightarrow \sqrt{\varepsilon} f$ and γ is not scaled.

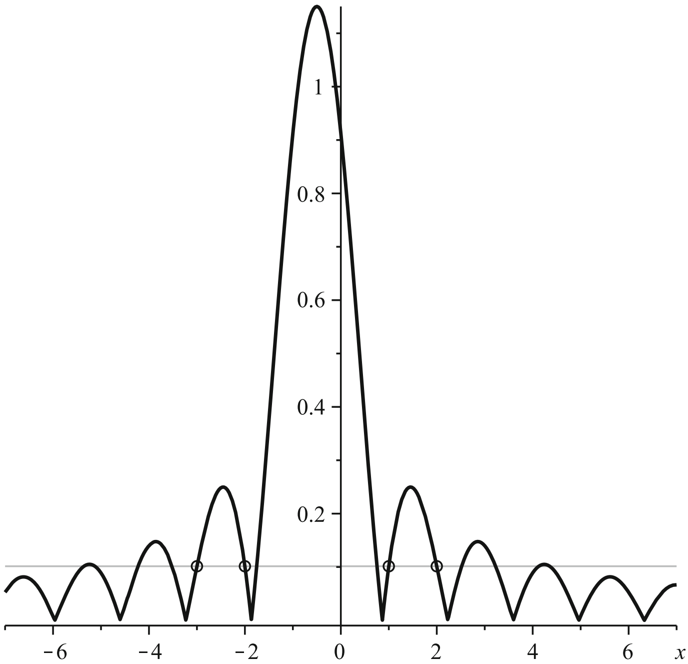

Fig. 2 The graph of functions $\left|\frac{\sin\frac{2x+1}{2}p}{2x+1}\right|$ and $\frac{\sqrt{5}}{9\sqrt{6}}$

More generally, if $r_1 = s_1 = m_1 - n_1$ and $k_1 = 2m_1 - 3n_1$ with $5m_1 > 9n_1$ then we obtain

$$\begin{aligned} ay + b - \tilde{c}x(x^2 + y^2) + O\left(\varepsilon^{\min\{m_1-n_1, 5m_1-9n_1\}}\right) &= 0 \\ -ax - \tilde{c}y(x^2 + y^2) + O\left(\varepsilon^{\min\{m_1-n_1, 5m_1-9n_1\}}\right) &= 0. \end{aligned} \tag{61}$$

We again obtain an equation similar to (45). So there is a unique solution of Eq. (61) for ε sufficiently small and $\gamma > 0$. Note that this solution of (37) is of order $O\left(\varepsilon^{\frac{m_1-n_1}{2m_1-3n_1}}\right)$, $\varepsilon > 0$ while $f \leftrightarrow \varepsilon^{\frac{m_1}{2m_1-3n_1}} f$ and $\gamma \leftrightarrow \varepsilon^{\frac{n_1}{2m_1-3n_1}}\gamma$. Of course, Theorem 3 can be also applied to (37) but the above scaling method helps to localize possible solutions.

2. Assumption (R1) does make sense. One can study double or higher resonances. E.g., let $\omega = \frac{2\sqrt{5}}{9}$, $\alpha = 6$, $p = 2\arctan\sqrt{5}$. Then Eq. (25) has two couples of solutions $\{1, -2\}$, $\{2, -3\}$ (see Fig. 2), since

$$\begin{aligned} \left|\frac{\sin\frac{2k_1+1}{2}p}{2k_1+1}\right| &= \frac{1}{3}\left|\sin(3\arctan\sqrt{5})\right| = \frac{\sqrt{5}}{9\sqrt{6}} = \frac{\omega}{2\sqrt{\alpha}}, \\ \left|\frac{\sin\frac{2k_2+1}{2}p}{2k_2+1}\right| &= \frac{1}{5}\left|\sin(5\arctan\sqrt{5})\right| = \frac{\sqrt{5}}{9\sqrt{6}} = \frac{\omega}{2\sqrt{\alpha}} \end{aligned}$$

for $k_1 = 1, k_2 = 2$.

5 Numerical Results

To illustrate the theoretical results obtained in the previous sections, we have solved the governing Eqs. (1) and (2) numerically. The nonlocal Eq. (33) is omitted here as its characteristics are rather similar to the local case. The advance-delay Eq. (2) is solved using a pseudo-spectral method. We express the solution U in a Fourier series

$$U(z) = \sum_{j=1}^{J} \left[A_j \cos\left(j\tilde{k}z\right) + B_j \sin\left(j\tilde{k}z\right)\right], \tag{62}$$

where $\tilde{k} = 2\pi/L$ and $-L/2 < z < L/2$. The Fourier coefficients A_j and B_j are then found by requiring the series to satisfy (2) at several collocation points. Hence, $2J$ collocation points are required. Note that in (62), we omit the zero harmonic $j = 0$ because it does not play any role, i.e. (2) is invariant under the transformation $U(z) \to U(z) + K$ for any constant K. Typically, we use $L = 4\pi$ and $J = 50$. The collocation points are chosen with uniform grid points. In the following, we only report the results for $\alpha = \beta = 1$, and $p = \pi$. Computations for different parameter values are performed, where we did not see any significant difference. To verify that (3) is satisfied by the numerically obtained solutions, we also follow $\max(|U(z+\pi) + U(z)|)$ at the collocation points. For all the results presented herein, the quantity is of order $\mathcal{O}\left(10^{-16}\right)$.

The stability of a solution obtained from (2) is determined numerically through calculating its Floquet multipliers, which are eigenvalues of the monodromy matrix. As a first-order system, the linearized equation of (1) is

$$\dot{u}_n = v_n, \tag{63}$$

$$\begin{aligned}\dot{v}_n = {} & \alpha\left(u_{n+1} + u_{n-1} - 2u_n\right) + 3\beta\left(U_{n+1}(t) - U_n(t)\right)^2\left(u_{n+1} - u_n\right) \\ & + 3\beta\left(U_{n-1}(t) - U_n(t)\right)^2\left(u_{n-1} - u_n\right) - \gamma v_n,\end{aligned} \tag{64}$$

where $U_n(t) = U(\omega t + np)$ (cf. (62)) is a periodic solution of (1). The linear system is integrated using a Runge-Kutta method of order four with periodic boundary conditions over the period $T = 2\pi/\omega$. The ith column of the monodromy matrix M is vector $[u_1(T), \ldots, u_N(T), v_1(T), \ldots, v_N(T)]^{\mathrm{t}}$, that corresponds to the initial condition $[u_1(0), \ldots, u_N(0), v_1(0), \ldots, v_N(0)]^{\mathrm{t}}$ equal to the ith column vector of the identity matrix I_{2N}. For the coupled ordinary differential equation system (1), one of the Floquet multipliers is always 1, which is also called the trivial Floquet multiplier. A periodic solution is asymptotically stable if all Floquet multipliers except the trivial Floquet multiplier are strictly smaller than one in modulus. The (in)stability result obtained from calculating the monodromy matrix is also confirmed by integrating (1).

From (12) (cf. Remark 1.3), for the parameter values above and $\gamma = 0$ the first resonance of the system is $\kappa = 1$, which corresponds to $\omega = 2$. In Fig. 3, we depict the bifurcation diagrams of periodic solutions for two values of $\omega > 2$. On the vertical axis is the oscillation amplitude of the periodic solutions as a function of driving amplitude f. When $\gamma = 0$ there is only one saddle-node bifurcation that occurs as we vary f as shown in dash-dotted lines for $\omega = 2.2$. Note that the system is symmetric with the transformation $f \to -f$, $U \to -U$. When γ is turned on to a non-zero value, the bifurcation curve interacts with its symmetry and at the intersection point an additional turning-point is formed, i.e. there is a curve-splitting-and-merging. Shown in dotted line is the approximation (31) where $2\sqrt{z} = U_{\max} - U_{\min}$.

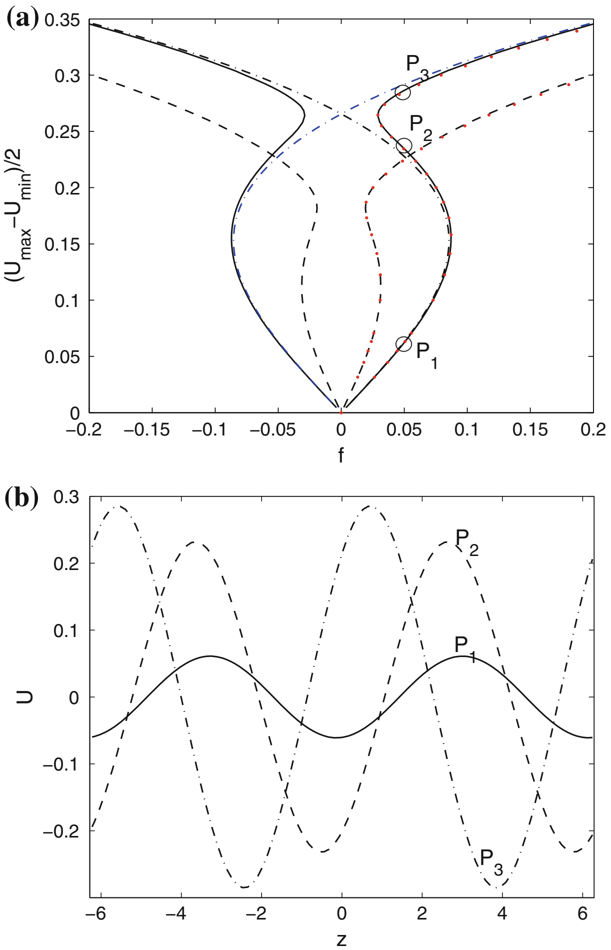

Fig. 3 **a** The oscillation amplitude of periodic solutions of (2) as a function of the driving amplitude f for $\omega = 2.1$ (*dashed*) and $\omega = 2.2$ (*solid*) with $\gamma = 0.05$. The *dash-dotted line* is the corresponding curve with $\omega = 2.2$ and $\gamma = 0$. **b** Solution profiles at the three points in (**a**)

In the second panel of the figure, we present the solution profile at three different points indicated in the first panel.

When ω is decreased toward the resonance $\omega = 2$, the critical driving force f for the occurrence of the first saddle-node bifurcation, i.e. the bifurcation that also exists when $\gamma = 0$, decreases towards zero. This is illustrated in panel (a) of Fig. 3 for two different values of ω. In the limit $\omega = 2$, we present in the top panel of Fig. 4 the existence curve of periodic solutions at the resonance. The numerics is in agreement with Theorem 6 that at the resonance $\kappa = 1$ there is a solution to the advance-delay equation for small driving force f. Using the Lyapunov-Schmidt reduction, the existence analysis requires nonzero damping γ because when $\gamma = 0$ the slope of the existence curve is singular at $f = 0$ as shown numerically in dashed line in the panel. As for the next resonances $\kappa = 3, 5, 7, \ldots$, the curve slopes are regular even when $\gamma = 0$ as depicted in the second panel of Fig. 4. This is also in agreement with Theorem 6 that there exist periodic solutions for small driving force f even when $\gamma = 0$.

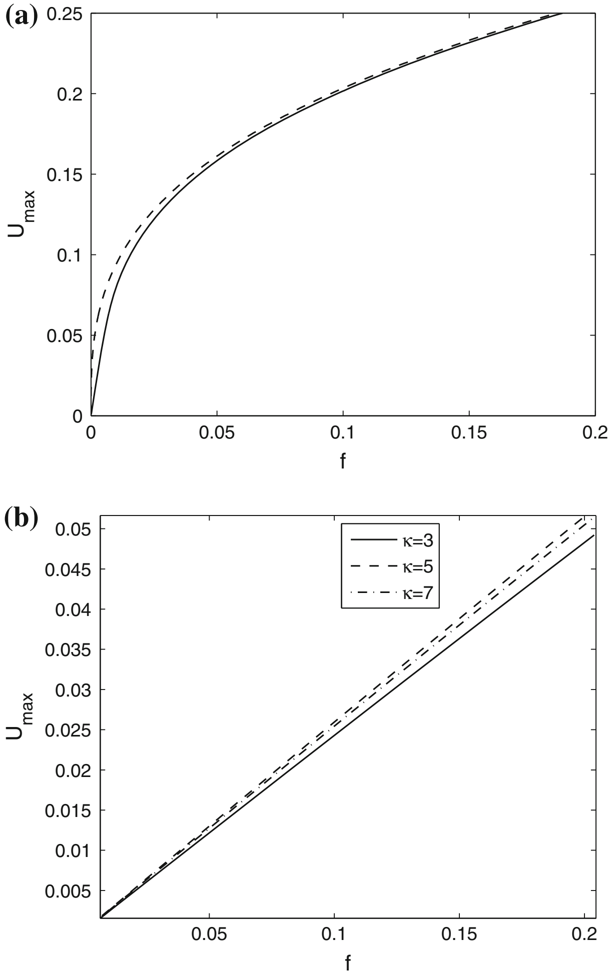

Fig. 4 Similar to the *top* panel of Fig. 3 for values of ω at the first few resonances. In panel **a**, $\omega = 2$ with $\gamma = 0.05$ and $\gamma = 0$ shown in *solid* and *dashed line*, respectively. In **b**, the next resonances as indicated in the legend with $\gamma = 0$

As for periodic solutions with driving frequency $\omega < 2$ satisfying (12), we found that the bifurcation curves are similar to those in the bottom panel of Fig. 4 and hence are not shown here.

After analysing the existence of periodic solutions numerically, we compute the stability of the solutions. It is important to note that the number of sites used to compute the Floquet multipliers influences the stability of the solutions [31]. Here, we used $N = 100$. Shown in the top panel of Fig. 5 is the distribution of the eigenvalues in the complex plane of the periodic solution P_1 in the second panel of Fig. 3. Using our numerics, upon increasing f we observed that there is a real eigenvalue bifurcating from the unit circle at the first saddle-node bifurcation. It is important to note that our observation here is different from the analysis of [25]. In [25], it was reported that the $\pi-$mode of the periodic solution becomes unstable before the saddle-node bifurcation at $f = f_{cr}^{int}$. We have also computed periodic solutions

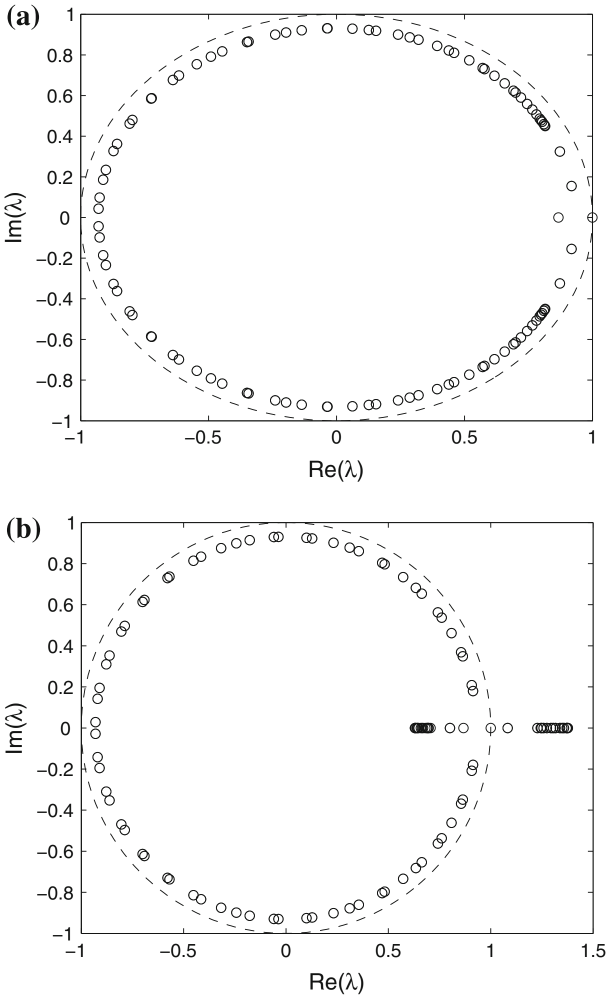

Fig. 5 The multipliers of the periodic solutions P_1 (**a**) and P_2 in Fig. 3 calculated using $N = 100$

for the same parameter values as in [25], but we still obtained the same scenario as mentioned above. It is not yet clear whether the number of sites we used, i.e. $N = 100$ as opposed to $N = 512$ in [25], is the reason for the different stability result. This is addressed for future work.

In the second panel of Fig. 5 we show the eigenvalue distribution of P_2, where we obtained that the periodic solution is exponentially unstable, i.e. all the unstable eigenvalues are real. From the first turning-point (see Fig. 3), as f decreases the solution picks up more unstable eigenvalues. The eigenvalue distribution of P_3 is similar to that of P_2 with more real unstable eigenvalues. Because only the most upper branch, which corresponds to unstable solutions, survives in the limit of the first resonance $\kappa = 1$, one would normally expect that when $\omega = 2$ all the solutions are unstable. This is indeed the case when $\gamma = 0$. For nonzero γ there is a stability window before a real eigenvalue leaves the unit circle and creates an instability to the solution. For $\gamma = 0.05$, the periodic solution is stable for $f \lessapprox 0.014$. For $\omega < 2$, there is a stability window of driving force in which periodic solutions are stable, even for $\gamma = 0$.

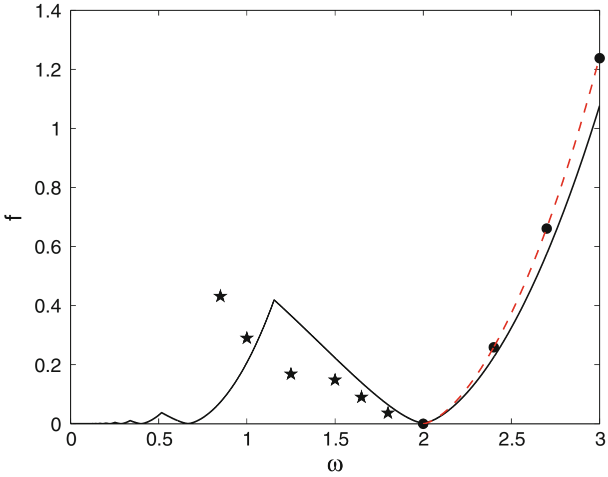

Fig. 6 Plot of Eq. (18). *Circles* are numerically computed points of saddle-node bifurcations and *stars* are critical points at which periodic solutions change stability. *Red dashed line* is f_- derived in [25]. Here, $\gamma = 0$ (Color figure online)

From dynamical systems, bifurcations occur when one or more Floquet multipliers cross the unit circle in the complex plane. For the saddle-node bifurcations above, we have seen that at the turning points there is an eigenvalue bifurcating from the real eigenvalue $\lambda = 1$. For $\omega < 2$, as there is no saddle-node bifurcation here, one possible bifurcation that occurs when a periodic solution becomes unstable is a pitchfork bifurcation. In both cases, at the bifurcation point the condition for uniqueness is degenerate. This may correspond to Theorems 1 and 2. Finally, in Fig. 6 we compare (18) and the numerically computed saddle-node points for $\omega > 2$ and the critical points of f at which periodic solutions change stability for $\omega < 2$. One can observe that the analytical and numerical results are close to each other in the neighbourhood of $\omega = 2$. As a comparison, we also plotted f_- derived in [25,26] shown as dashed line, see (31).

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