



# The stationary sine-Gordon equation on metric graphs: Exact analytical solutions for simple topologies

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## ABSTRACT

We consider the stationary sine-Gordon equation on metric graphs with simple topologies. Exact analytical solutions are obtained for different vertex boundary conditions. It is shown that the method can be extended for tree and other simple graph topologies. Applications of the obtained results to branched planar Josephson junctions and Josephson junctions with tricrystal boundaries are discussed.

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## 1. Introduction

Nonlinear wave equations have numerous applications in different topics of physics and natural sciences (see, e.g., [1–6]). Recently they have attracted a lot attentions in the context of soliton transports in networks and branched structures [7–19]. Wave dynamics in networks can be modelled by nonlinear evolution equations on metric graphs. This fact greatly facilitates the study of soliton transports in branched systems. Metric graph is a system of bonds which are assigned a length and connected at the vertices according to a rule, called “topology of a graph”. Solitons and other nonlinear waves in branched systems appear in different systems of condensed matter, polymers, optics, neuroscience, DNA and many others.

In condensed matters, very important branched systems where solitons can appear are the Josephson junction networks [20]–[21]. The phase difference in a Josephson junction obeys the sine-Gordon equation [22]. Josephson junction networks can therefore be effectively modelled by the sine-Gordon equation on metric graphs. The early treatment of superconductor networks consisting of Josephson junctions meeting at one point dated back to [23, 24]. An interesting realization of Josephson junction networks at

tricrystal boundaries was discussed earlier in [25], which inspired later detailed study of the problem using the sine-Gordon equation on networks in [17,26,27]. Discrete sine-Gordon equations were also used in [20,21,28] to describe different networks of Josephson junctions having several junctions on each wire of a network. Recently, a 2D sine-Gordon equation on networks was studied by considering *Y* and *T* junctions [18]. Discrete sine-Gordon equations on networks were also considered in [29].

In this paper we address the problem of stationary sine-Gordon equations on metric graphs by focusing on exact analytical solutions for simple graph topologies. Such a one-dimensional, stationary sine-Gordon equation describes, for instance, the transverse component of the phase difference in a 2D Josephson junction in a constant magnetic field. The derivative of the phase difference presents the local magnetic field in the system [30–32]. Planar Josephson junctions were studied in [31,32] on the basis of solutions of the stationary sine-Gordon equation on a finite interval. Here, we use a similar approach to solve the stationary sine-Gordon equation on metric graphs. Two types of vertex boundary conditions are considered providing different conservation laws, continuity of the wave function and its derivatives. The model proposed in this work can be used to describe static solitons in 2D Josephson junctions interacting with constant magnetic field [31, 32]. The results are then extended for metric tree graphs consisting of finite bonds. The study can be generalized to other simple

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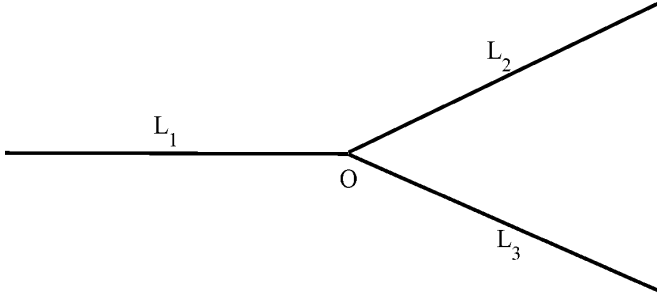


Fig. 1. Sketch of a metric star graph.  $L_j$  is the length of the  $j$ th bond with  $j = 1, 2, 3$ .

graph topologies, which can be constructed using star and loop graphs.

This paper is organized as follows. In the next section we give a formulation of the problem by deriving the boundary conditions for the a star graph and provide the exact analytical solutions for special cases. In Section 3, we extend the computational procedures to metric tree graphs. In Section 4, we explore the stability of the obtained solutions. Finally, Section 5 presents some concluding remarks.

**2. Vertex boundary conditions and exact solutions for star graph**

The static sine-Gordon equation on a metric graph presented in Fig. 1 can be written as

$$\frac{d^2}{dx^2} \phi_j = \frac{1}{\lambda_j^2} \sin(\phi_j), \quad 0 < x < L_j, \tag{1}$$

where the wave function  $\phi_j$  is assigned to each bond of the graph and  $j = 1, 2, 3$  is the bond number. For wave equations on networks, the connections of the network wires at the vertices are provided by the vertex boundary conditions. In case of linear wave equations, the underlying constraint to derive vertex boundary conditions is the self-adjointness of the problem [33,34]. However, for nonlinear case one should use different conservation laws [7,9, 18]. Here, for the stationary sine-Gordon equation we impose two types of boundary conditions. The first type provides the continuity of the ‘weighted’ wave function derivatives

$$\lambda_1 \left. \frac{d\phi_1}{dx} \right|_{x=0} = \lambda_2 \left. \frac{d\phi_2}{dx} \right|_{x=0} = \lambda_3 \left. \frac{d\phi_3}{dx} \right|_{x=0} \tag{2}$$

and conservation of the magnetic self-field flux at the vertex, which are given as

$$\lambda_1 \phi_1|_{x=0} + \lambda_2 \phi_2|_{x=0} + \lambda_3 \phi_3|_{x=0} = 0. \tag{3}$$

The second type of vertex conditions has the form of wave function continuity at the vertex

$$\phi_1|_{x=0} = \phi_2|_{x=0} = \phi_3|_{x=0}, \tag{4}$$

and conservation of the applied magnetic flux at the vertex, which is given as

$$\lambda_1^2 \left. \frac{d\phi_1}{dx} \right|_{x=0} + \lambda_2^2 \left. \frac{d\phi_2}{dx} \right|_{x=0} + \lambda_3^2 \left. \frac{d\phi_3}{dx} \right|_{x=0} = 0. \tag{5}$$

In terms of Josephson junction networks the vertex conditions (3) and (5) imply Kirchhoff rules for self-induced and external magnetic field fluxes. The boundary condition at the end of each bond is given by

$$\left. \frac{d\phi_j}{dx} \right|_{x=L_j} = 2H_j, \tag{6}$$

with  $H_j$  being the homogeneous external magnetic field applied along the  $j$ th bond. Such boundary conditions may appear in, e.g., branched graphene nanoribbons [35,36]. Since strained graphene creates pseudo magnetic field [37], in this way it is possible to obtain magnetic fields of different strengths along the different junctions when they have different strains.

The boundary conditions (2)–(3) are consistent with other models of Josephson junction networks previously studied theoretically in [17,25,31,32] as well as experimentally in [38–40]. Detailed derivation of these boundary conditions are resented in Appendix A. Exact solutions of Eq. (1) have been obtained earlier in [31,32,41] for the second boundary conditions on a finite interval. Here, we use an approach similar to that in [31,32] to obtain exact analytical solutions of Eq. (1) for the boundary conditions (2)–(6).

*2.1. Solution of type I*

Our purpose is to obtain exact analytical solutions of the problem given by Eqs. (1)–(6). A solution of Eq. (1) without boundary conditions can be written as [31,32]

$$\phi_j^{(\pm)}(x) = (2n_j + 1)\pi \pm 2 \arcsin \left\{ k_j \operatorname{sn} \left[ \frac{x - x_{0,j}^{(\pm)}}{\lambda_j}, k_j \right] \right\} \tag{7}$$

where  $k_j$  and  $x_{0,j}^{(\pm)}$  are integration constants and  $\operatorname{sn}$  is Jacobi’s elliptic function. Depending on the value of  $k_j$  that is determined by the constraint  $|H_j \lambda_j| \leq |k_j| \leq 1$ , we refer to the solution as solution of type 1 [31]. Taking into account that

$$\frac{d\phi_j^{(\pm)}}{dx} = \pm \frac{2k_j}{\lambda_j} \operatorname{cn} \left[ \frac{x - x_{0,j}^{(\pm)}}{\lambda_j}, k_j \right], \tag{8}$$

from boundary condition (6) we have

$$x_{0,j}^{(\pm)} = L_j - \lambda_j F \left[ \arccos \left( \pm \frac{H_j \lambda_j}{k_j} \right), k_j \right]. \tag{9}$$

Here,  $\operatorname{cn}$  is Jacobi’s elliptic function [42] and  $F(\varphi, k)$  is the elliptic integral of the first kind [42]. Then solution of type 1 of the sine-Gordon equation on a metric star graph with the boundary conditions (2)–(3) can be written as

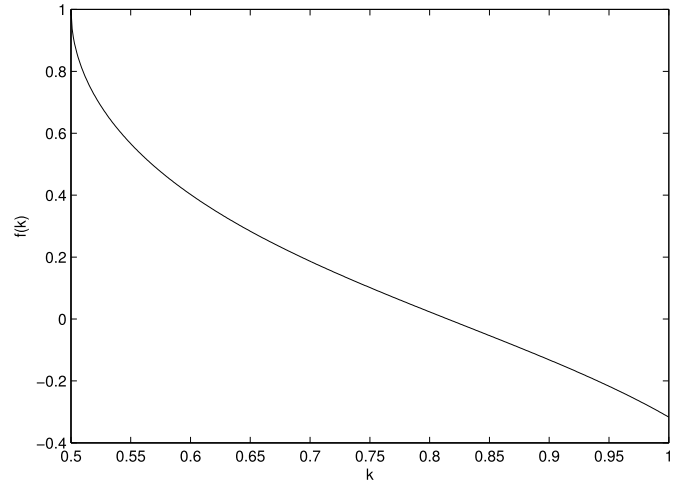
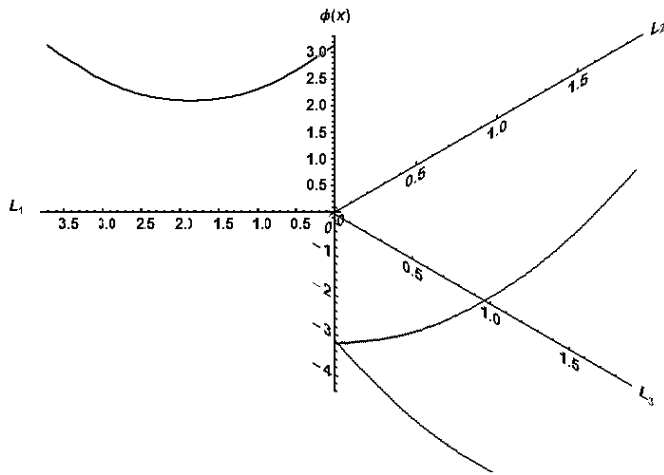
$$\phi_j^{(\pm)}(x) = (2n_j + 1)\pi \pm \pm 2 \arcsin \left\{ k_j \operatorname{sn} \left[ \frac{x - L_j}{\lambda_j} + F \left[ \arccos \left( \pm \frac{H_j \lambda_j}{k_j} \right), k_j \right], k_j \right] \right\}.$$

The vertex boundary conditions (2) and (3) lead to the following system of transcendental equations for finding  $k_j$ :

$$\sum_{j=1}^3 \lambda_j \arcsin \left\{ k_j \operatorname{sn} \left[ \frac{L_j}{\lambda_j} - F \left[ \arccos \left( \pm \frac{H_j \lambda_j}{k_j} \right), k_j \right], k_j \right] \right\} = \pm \frac{1}{2} \sum_{j=1}^3 (2n_j + 1)\pi \lambda_j, \tag{10}$$

$$k_1 \operatorname{cn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] = k_2 \operatorname{cn} \left[ \frac{L_2}{\lambda_2} - F \left[ \arccos \left( \pm \frac{H_2 \lambda_2}{k_2} \right), k_2 \right], k_2 \right], \tag{11}$$

$$k_1 \operatorname{cn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] = k_3 \operatorname{cn} \left[ \frac{L_3}{\lambda_3} - F \left[ \arccos \left( \pm \frac{H_3 \lambda_3}{k_3} \right), k_3 \right], k_3 \right]. \tag{12}$$



**Fig. 2.** Solution of type 1 of Eq. (1) for case 1, plotted for  $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5, m = 1, H_1 = 0.5, H_2 = 1, H_3 = 1, H = 0.5, k = H$ .

**Fig. 3.** Plot of the function  $f(k)$  for  $H = 0.5\lambda_1 = 0.5, \lambda_2 = 0.2, \lambda_3 = 0.3, m = 1, p = 1, n_1 = -1, n_2 = n_3 = 0$ , which shows the existence of a root of Eq. (13).

It is clear that if this system has roots, then our problem has solutions.

Here, we consider exact analytical solutions of this system for two special cases.

Case 1 is given by the relations

$$\lambda_1 = \lambda_2 + \lambda_3, \frac{L_j}{\lambda_j} = 2mK(k_j), m \in \mathbf{N},$$

$$H_1\lambda_1 = H_2\lambda_2 = H_3\lambda_3 = H > 0,$$

$$n_1 = -n, n_2 = n_3 = n - 1, n \in \mathbf{Z}.$$

From Eqs. (10)–(12), we have

$$k_1 = k_2 = k_3 = k$$

and

$$g^{(\pm)}(k) \equiv (-1)^{m+1}k\sqrt{1 - \left(\frac{H}{k}\right)^2} = 0,$$

which gives

$$k = \pm H.$$

In Fig. 2 the solution corresponding to this special case is plotted.

Case 2 corresponds to the constraints

$$\lambda_1 = \lambda_2 + \lambda_3, \frac{L_j}{\lambda_j} = p + 4m_jK(k_j), m_j \in \mathbf{N} \cup \{0\},$$

$$H_1\lambda_1 = H_2\lambda_2 = H_3\lambda_3 = H > 0,$$

$$n_1 = -n, n_2 = n_3 = n - 1, n \in \mathbf{Z},$$

where  $0 < p < F[\arccos(H), 1]$ . Then from Eqs. (10)–(12) we have

$$k_1 = k_2 = k_3 = k$$

and

$$f^{(\pm)}(k) \equiv p - F\left[\arccos\left(\pm\frac{H}{k}\right), k\right] = 0. \tag{13}$$

Since  $f^{(\pm)}(\pm H) > 0$ ,  $f^{(\pm)}(\pm 1) < 0$  and the function  $f^{(+)}(k)$  ( $f^{(-)}(k)$ ) is continuous on interval  $[H; 1]$  ( $[-1; -H]$ ), the system has at least one root. This can be seen in Fig. 3, where the function  $f(k)$  is plotted.

The treatment of the problem for the second type vertex boundary conditions, (4) and (5) is similar to the above. The solution of Eq. (1) fulfilling these boundary conditions is the same

as that for the conditions given by Eqs. (2) and (3). However, the system of transcendental equations for determining  $k_j$  are slightly different from those for the first vertex conditions, i.e.

$$\begin{aligned} & (2n_1 + 1)\pi \pm \\ & \pm 2\arcsin\left\{k_1\operatorname{sn}\left[\frac{L_1}{\lambda_1} - F\left[\arccos\left(\mp\frac{H_1\lambda_1}{k_1}\right), k_1\right], k_1\right]\right\} = \\ & = (2n_2 + 1)\pi \pm \\ & \pm 2\arcsin\left\{k_2\operatorname{sn}\left[\frac{L_2}{\lambda_2} - F\left[\arccos\left(\mp\frac{H_2\lambda_2}{k_2}\right), k_2\right], k_2\right]\right\}, \end{aligned} \tag{14}$$

$$\begin{aligned} & (2n_1 + 1)\pi \pm \\ & \pm 2\arcsin\left\{k_1\operatorname{sn}\left[\frac{L_1}{\lambda_1} - F\left[\arccos\left(\mp\frac{H_1\lambda_1}{k_1}\right), k_1\right], k_1\right]\right\} = \\ & = (2n_3 + 1)\pi \pm \\ & \pm 2\arcsin\left\{k_3\operatorname{sn}\left[\frac{L_3}{\lambda_3} - F\left[\arccos\left(\mp\frac{H_3\lambda_3}{k_3}\right), k_3\right], k_3\right]\right\}, \end{aligned} \tag{15}$$

$$\begin{aligned} & k_1\lambda_1\operatorname{cn}\left[\frac{L_1}{\lambda_1} - F\left[\arccos\left(\pm\frac{H_1\lambda_1}{k_1}\right), k_1\right], k_1\right] + \\ & + k_2\lambda_2\operatorname{cn}\left[\frac{L_2}{\lambda_2} - F\left[\arccos\left(\pm\frac{H_2\lambda_2}{k_2}\right), k_2\right], k_2\right] + \\ & + k_3\lambda_3\operatorname{cn}\left[\frac{L_3}{\lambda_3} - F\left[\arccos\left(\pm\frac{H_3\lambda_3}{k_3}\right), k_3\right], k_3\right] = 0. \end{aligned} \tag{16}$$

Following the discussion for the first vertex conditions (2) and (3), one can also solve this system for two special cases. However, we obtain that the corresponding first case can be solved analytically, while the second one has to be solved numerically.

### 2.2. Solutions of type II

Solutions of type II for Eq. (1) are given by

$$\phi_j^{(\pm)}(x) = \pi(2n_j + 1) \pm 2\operatorname{am}\left(\frac{x - x_{0,j}^{(\pm)}}{\lambda_j k_j}, k_j\right), \tag{17}$$

and defined by the constraint

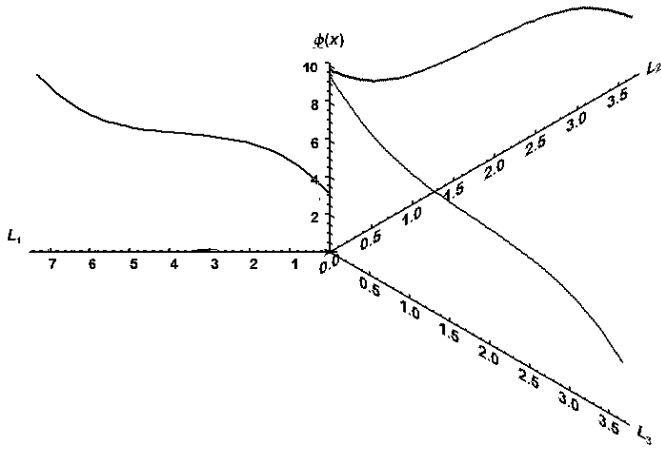


Fig. 4. Solution of type 1 of Eq. (1) for case 1, plotted for  $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = 0.5, m = 1, H_1 = 1.01, H_2 = 2.02, H_3 = 2.02, H = 1.01, k = \frac{1}{H}$ .

$$\frac{1}{\sqrt{1 + H_j^2 \lambda_j^2}} \leq |k_j| \leq \frac{1}{|H_j \lambda_j|}.$$

For the derivative of this solution, we have

$$\frac{d\phi_j^{(\pm)}(x)}{dx} = \pm \frac{2}{\lambda_j k_j} \operatorname{dn} \left( \frac{x - x_{0,j}^{(\pm)}}{\lambda_j k_j}, k_j \right). \quad (18)$$

Inserting this derivative into the boundary condition (6) yields

$$x_{0,j}^{(\pm)} = L_j \mp \lambda_j k_j F \left( \arcsin \frac{\sqrt{1 - H_j^2 \lambda_j^2 k_j^2}}{k_j}, k_j \right). \quad (19)$$

Eqs. (17)–(19) together with the boundary conditions (2), (3) lead to

$$\sum_{j=1}^3 \lambda_j \operatorname{am} \left[ \frac{L_j}{\lambda_j k_j} \mp F \left( \arcsin \frac{\sqrt{1 - H_j^2 \lambda_j^2 k_j^2}}{k_j}, k_j \right), k_j \right] = \pm \frac{1}{2} \sum_{j=1}^3 (2n_j + 1) \pi \lambda_j, \quad (20)$$

$$\frac{1}{k_1} \operatorname{dn} \left[ \frac{L_1}{\lambda_1 k_1} \mp F \left( \arcsin \frac{\sqrt{1 - H_1^2 \lambda_1^2 k_1^2}}{k_1}, k_1 \right), k_1 \right] = \frac{1}{k_2} \operatorname{dn} \left[ \frac{L_2}{\lambda_2 k_2} \mp F \left( \arcsin \frac{\sqrt{1 - H_2^2 \lambda_2^2 k_2^2}}{k_2}, k_2 \right), k_2 \right], \quad (21)$$

$$\frac{1}{k_1} \operatorname{dn} \left[ \frac{L_1}{\lambda_1 k_1} \mp F \left( \arcsin \frac{\sqrt{1 - H_1^2 \lambda_1^2 k_1^2}}{k_1}, k_1 \right), k_1 \right] = \frac{1}{k_3} \operatorname{dn} \left[ \frac{L_3}{\lambda_3 k_3} \mp F \left( \arcsin \frac{\sqrt{1 - H_3^2 \lambda_3^2 k_3^2}}{k_3}, k_3 \right), k_3 \right]. \quad (22)$$

Again we show the existence of solutions of Eqs. (20)–(22) for two special cases.

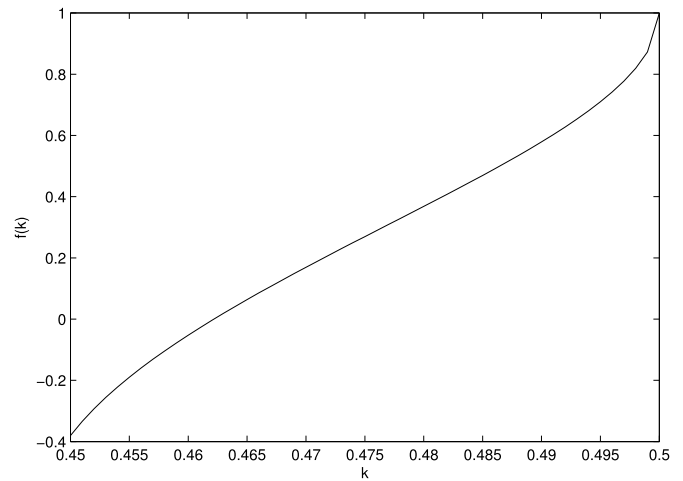


Fig. 5. Plot of the function  $f(k)$  for  $H = 2\lambda_1 = 0.5, \lambda_2 = 0.2, \lambda_3 = 0.3, m_1 = 2, m_2 = m_3 = 1, p = 1, n_1 = n_2 = n_3 = 1$ , showing the existence of a root of Eq. (25).

For case 1, which corresponds to the relations

$$\sum_{j=1}^3 (2n_j \mp 2m + 1) \lambda_j = 0, \quad m \in \mathbf{N}$$

$$\frac{L_j}{\lambda_j} = 2mk_j K(k_j), \quad m \in \mathbf{N},$$

$$H_1 \lambda_1 = H_2 \lambda_2 = H_3 \lambda_3 = H > 1, \quad (23)$$

from Eqs. (20)–(22), we have

$$g^{(\pm)}(k) \equiv \operatorname{am} \left[ F \left( \arcsin \frac{\sqrt{1 - H^2 k^2}}{k}, k \right), k \right] = 0. \quad (24)$$

Then, Eq. (24) gives the following solution for the system of transcendental equations (20)–(22):

$$k = \pm \frac{1}{H}.$$

Fig. 4 presents plot of the solution of Eq. (1) corresponding to this special case.

For case 2, which is defined by the conditions

$$\frac{L_j}{\lambda_j} = k_j (p + 2m_j K(k_j)), \quad m_j \in \mathbf{N} \cup \{0\},$$

$$H_1 \lambda_1 = H_2 \lambda_2 = H_3 \lambda_3 = H > 1,$$

$$\sum_{j=1}^3 (2n_j \mp 2m_j + 1) \lambda_j = 0,$$

where  $0 < p < K \left( \frac{1}{\sqrt{1+H^2}} \right)$ , Eqs. (26)–(28) yield

$$k_1 = k_2 = k_3 = k,$$

that leads to

$$f^{(\pm)}(k) \equiv p \mp F \left[ \arcsin \frac{\sqrt{1 - H^2 k^2}}{k}, k \right] = 0. \quad (25)$$

Since  $f^{(\pm)}(\pm \frac{1}{H}) > 0, f^{(\pm)}(\pm \frac{1}{\sqrt{1+H^2}}) < 0$  and the function  $f^{(+)}(k)$  ( $f^{(-)}(k)$ ) is continuous on interval  $[\frac{1}{\sqrt{1+H^2}}; \frac{1}{H}]$  ( $[-\frac{1}{H}; -\frac{1}{\sqrt{1+H^2}}]$ ), it has at least one root on this interval. Fig. 5 with the plot of  $f(k)$  clearly shows that.

Again, similarly to the above, solutions of type II can be obtained for the second vertex boundary conditions, (4) and (5), which have the same wave function as that for the vertex conditions given by Eqs. (2) and (3), but with a different transcendental system given by

$$\begin{aligned} & (2n_1 + 1)\pi \\ & \pm 2am \left[ \frac{L_1}{\lambda_1 k_1} \pm F \left[ \arcsin \frac{\sqrt{1 - H_1^2 \lambda_1^2 k_1^2}}{k_1}, k_1 \right], k_1 \right] = \\ & = (2n_2 + 1)\pi \\ & \pm 2am \left[ \frac{L_2}{\lambda_2 k_2} \pm F \left[ \arcsin \frac{\sqrt{1 - H_2^2 \lambda_2^2 k_2^2}}{k_2}, k_2 \right], k_2 \right], \end{aligned} \tag{26}$$

$$\begin{aligned} & (2n_1 + 1)\pi \\ & \pm 2am \left[ \frac{L_1}{\lambda_1 k_1} \pm F \left[ \arcsin \frac{\sqrt{1 - H_1^2 \lambda_1^2 k_1^2}}{k_1}, k_1 \right], k_1 \right] = \\ & = (2n_3 + 1)\pi \\ & \pm 2am \left[ \frac{L_3}{\lambda_3 k_3} \pm F \left[ \arcsin \frac{\sqrt{1 - H_3^2 \lambda_3^2 k_3^2}}{k_3}, k_3 \right], k_3 \right], \end{aligned} \tag{27}$$

$$\begin{aligned} & \frac{\lambda_1}{k_1} dn \left[ \frac{L_1}{\lambda_1 k_1} \pm F \left[ \arcsin \frac{\sqrt{1 - H_1^2 \lambda_1^2 k_1^2}}{k_1}, k_1 \right], k_1 \right] + \\ & + \frac{\lambda_2}{k_2} dn \left[ \frac{L_2}{\lambda_2 k_2} \pm F \left[ \arcsin \frac{\sqrt{1 - H_2^2 \lambda_2^2 k_2^2}}{k_2}, k_2 \right], k_2 \right] + \\ & + \frac{\lambda_3}{k_3} dn \left[ \frac{L_3}{\lambda_3 k_3} \pm F \left[ \arcsin \frac{\sqrt{1 - H_3^2 \lambda_3^2 k_3^2}}{k_3}, k_3 \right], k_3 \right] = 0. \end{aligned} \tag{28}$$

This system can be solved similarly to the case of vertex conditions given by Eqs. (4) and (5).

It is important to note that in the limit of infinitely long bonds, solutions of type I and type II considered above do not become the soliton-like solutions considered in [17]. Due to our nonzero Neumann boundary conditions, the solution along each junction is either part of the ‘oscillating’ or the ‘rotating’ solution of the sine-Gordon equation (1).

### 3. Applications of the method in tree graphs

The procedure presented in Section 2 above can also be applied to other simple topologies, such as tree graphs, loops and their combinations. Here, we briefly demonstrate this for the tree graph presented in Fig. 6, which consists of three subgraphs  $b_1, (b_{1i}), (b_{1ij})$ , with  $i, j$  running over each bond of the subgraphs. On each bond  $b_1, b_{1i}, b_{1ij}$  with the lengths  $L_1, L_{1i} - L_1, L_{1ij} - L_{1i}$ , respectively, we have the stationary sine-Gordon equation (1). The vertex boundary conditions can be written similarly to those in Eqs. (2) and (3). Then we have

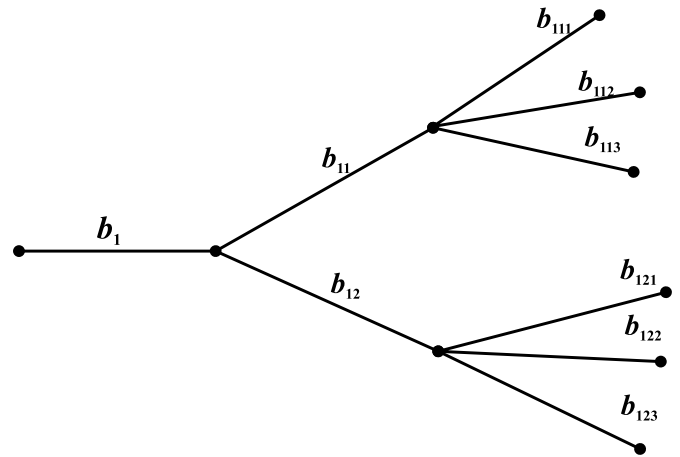


Fig. 6. A metric tree graph.

$$\begin{aligned} \frac{d\phi_1}{dx} \Big|_{x=0} &= 2H_1, \quad \frac{d\phi_{1ij}}{dx} \Big|_{x=L_{1ij}} = 2H_{1ij}, \quad i = 1, 2, \quad j = 1, 2, 3, \\ \lambda_1 \frac{d\phi_1}{dx} \Big|_{x=L_1} &= \lambda_{1i} \frac{d\phi_{1i}}{dx} \Big|_{x=L_i}, \quad i = 1, 2, \\ \lambda_{1i} \frac{d\phi_{1i}}{dx} \Big|_{x=L_{1i}} &= \lambda_{1ij} \frac{d\phi_{1ij}}{dx} \Big|_{x=L_{1ij}}, \quad i = 1, 2, \quad j = 1, 2, 3, \\ \lambda_1 \phi_1 \Big|_{x=L_1} + \lambda_{11} \phi_{11} \Big|_{x=L_1} + \lambda_{12} \phi_{12} \Big|_{x=L_1} &= 0, \\ \lambda_{1i} \phi_{1i} \Big|_{x=L_{1i}} + \sum_{j=1}^3 \lambda_{1ij} \phi_{1ij} \Big|_{x=L_{1ij}} &= 0, \quad i = 1, 2. \end{aligned}$$

Solutions of type 1 and 2 of Eq. (1) are defined similarly to those for star graphs and can be written as

$$\begin{aligned} \phi_b^{(\pm)}(x) &= (2n_b + 1)\pi \pm 2 \arcsin \left\{ k_b \operatorname{sn} \left[ \frac{x - x_{0,b}^{(\pm)}}{\lambda_b}, k_b \right] \right\}, \\ \phi_b^{(\pm)}(x) &= (2n_b + 1)\pi \pm 2 \operatorname{am} \left( \frac{x - x_{0,b}^{(\pm)}}{\lambda_b k_b}, k_b \right). \end{aligned}$$

Requiring these solutions to satisfy the boundary conditions leads to a system of transcendental equations for finding  $k_b$ . Again, exact solutions of this system can be obtained for two special cases. However, unlike the case of star graphs, for tree graphs different bonds may have different type of solutions, e.g., one bond can have a solution of type 1, while for the others it is possible to obtain the solution of type 2.

From the vertex boundary conditions for solution of type 1 we have the following system of transcendental equations

$$\begin{aligned} & \lambda_1 \arcsin \left\{ k_1 \operatorname{sn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] \right\} + \\ & \lambda_{11} \arcsin \left\{ k_{11} \operatorname{sn} \left[ \frac{L_1 - x_{0,11}^{(\pm)}}{\lambda_{11}}, k_{11} \right] \right\} + \\ & \lambda_{12} \arcsin \left\{ k_{12} \operatorname{sn} \left[ \frac{L_1 - x_{0,12}^{(\pm)}}{\lambda_{12}}, k_{12} \right] \right\} \\ & = \mp \frac{\pi}{2} [(2n_1 + 1)\lambda_1 + (2n_{11} + 1)\lambda_{11} + (2n_{12} + 1)\lambda_{12}], \tag{29} \\ & \lambda_{1i} \arcsin \left\{ k_{1i} \operatorname{sn} \left[ \frac{L_{1i} - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right] \right\} + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^3 \lambda_{1ij} \arcsin \left\{ k_{1ij} \operatorname{sn} \left[ \frac{L_{1i} - L_{1ij}}{\lambda_{1ij}} + \right. \right. \\
 & \left. \left. + F \left[ \arccos \left( \pm \frac{H_{1ij} \lambda_{1ij}}{k_{1ij}} \right), k_{1ij} \right], k_{1ij} \right] \right\} \\
 & = \mp \frac{\pi}{2} \left[ (2n_{1i} + 1) \lambda_{1i} + \sum_{j=1}^3 (2n_{1ij} + 1) \lambda_{1ij} \right], \tag{30}
 \end{aligned}$$

$$\begin{aligned}
 & k_1 \operatorname{cn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] \\
 & = k_{1i} \operatorname{cn} \left[ \frac{L_1 - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right], \tag{31}
 \end{aligned}$$

$$\begin{aligned}
 & k_{1i} \operatorname{cn} \left[ \frac{L_1 - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right] \\
 & = k_1 \operatorname{cn} \left[ \frac{L_{1i} - L_{1ij}}{\lambda_{1ij}} + F \left[ \arccos \left( \pm \frac{H_{1ij} \lambda_{1ij}}{k_{1ij}} \right), k_{1ij} \right], k_{1ij} \right], \tag{32}
 \end{aligned}$$

where  $i = 1, 2, j = 1, 2, 3$ . Again, as in the case of tree graph, one can consider two cases. Choosing  $x_{0,1i}^{(\pm)} = \frac{1}{2}(L_{1i} + L_1)$  for case 1 we have

$$\begin{aligned}
 & \lambda_1 + \lambda_{11} + \lambda_{12} = 0, \lambda_{1i} + \sum_{j=1}^3 \lambda_{1ij} = 0, \\
 & \frac{L_1}{\lambda_1} = 2mK(k_1), \frac{L_{1ij} - L_{1i}}{\lambda_{1ij}} = 2mK(k_{1ij}), \\
 & \frac{L_{1i} - L_1}{2\lambda_{1i}} = 2mK(k_{1i}) - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_{1i}} \right), k_{1i} \right], \\
 & H_1 \lambda_1 = H_{1ij} \lambda_{1ij} = H > 0, \\
 & n_1 = n_{1i} = n_{1ij} = n, n \in \mathbf{Z}, \tag{33}
 \end{aligned}$$

where  $i = 1, 2, j = 1, 2, 3, m \in \mathbf{Z} \setminus \{0\}$ . Then simplifying the above system of transcendental equations (29)–(32) will yield

$$k_b = k, g^{(\pm)}(k) \equiv (-1)^{m+1} k \sqrt{1 - \left(\frac{H}{k}\right)^2} = 0,$$

which together with Eq. (33) gives  $k = \pm H$ .

For case 2 we have

$$\begin{aligned}
 & \lambda_1 + \lambda_{11} + \lambda_{12} = 0, \lambda_{1i} + \sum_{j=1}^3 \lambda_{1ij} = 0, \\
 & \frac{L_1}{\lambda_1} = p + 4m_1K(k_1), \frac{L_{1ij} - L_{1i}}{\lambda_{1ij}} = p + 4m_{1ij}K(k_{1ij}), \\
 & \frac{L_{1i} - L_1}{2\lambda_{1i}} = p + 4m_{1i}K(k_{1i}) - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_{1i}} \right), k_{1i} \right], \\
 & H_1 \lambda_1 = H_{1ij} \lambda_{1ij} = H > 0, \\
 & n_1 = n_{1i} = n_{1ij} = n, n \in \mathbf{Z},
 \end{aligned}$$

where  $0 < p < F[\arccos(H), 1]$  and  $i = 1, 2, j = 1, 2, 3, m \in \mathbf{Z}$ . For this case the solution of Eqs. (29)–(32) can be written as

$$k_b = k, f^{(\pm)}(k) \equiv p - F \left[ \arccos \left( \pm \frac{H}{k} \right), k \right].$$

Since  $f^{(\pm)}(\pm \frac{1}{H}) > 0, f^{(\pm)}(\pm 1) < 0$  and the function  $f^{(+)}(k)$  ( $f^{(-)}(k)$ ) is continuous on interval  $[\frac{1}{H}; 1]$  ( $[-1; -\frac{1}{H}]$ ), it has at least one root on this interval. Similarly, one can obtain also the solutions of type 2.

Applying the above prescriptions, one can also obtain the corresponding solutions of Eq. (1) for the boundary conditions (4) and (5). This leads to the same solution (wave function) as that for the vertex boundary conditions (2) and (3). However, the transcendental equations for finding  $k_j$  are different and are given as

$$\begin{aligned}
 & (2n_1 + 1)\pi \pm \\
 & \pm \arcsin \left\{ k_1 \operatorname{sn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] \right\} = \\
 & = (2n_{1i} + 1)\pi \pm \arcsin \left\{ k_{1i} \operatorname{sn} \left[ \frac{L_1 - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right] \right\}, \tag{34}
 \end{aligned}$$

$$\begin{aligned}
 & (2n_{1i} + 1)\pi \pm \arcsin \left\{ k_{1i} \operatorname{sn} \left[ \frac{L_{1i} - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right] \right\} = \\
 & (2n_{1ij} + 1)\pi \pm \arcsin \left\{ k_{1ij} \operatorname{sn} \left[ \frac{L_{1i} - L_{1ij}}{\lambda_{1ij}} + \right. \right. \\
 & \left. \left. F \left[ \arccos \left( \pm \frac{H_{1ij} \lambda_{1ij}}{k_{1ij}} \right), k_{1ij} \right], k_{1ij} \right] \right\}, \tag{35}
 \end{aligned}$$

$$\begin{aligned}
 & \lambda_1 k_1 \operatorname{cn} \left[ \frac{L_1}{\lambda_1} - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_1} \right), k_1 \right], k_1 \right] = \\
 & \lambda_{11} k_{11} \operatorname{cn} \left[ \frac{L_1 - x_{0,11}^{(\pm)}}{\lambda_{11}}, k_{11} \right] + \lambda_{12} k_{12} \operatorname{cn} \left[ \frac{L_1 - x_{0,12}^{(\pm)}}{\lambda_{12}}, k_{12} \right] \tag{36}
 \end{aligned}$$

$$\begin{aligned}
 & \lambda_{1i} k_{1i} \operatorname{cn} \left[ \frac{L_{1i} - x_{0,1i}^{(\pm)}}{\lambda_{1i}}, k_{1i} \right] = \\
 & \sum_{j=1}^3 \lambda_{1ij} k_{1ij} \operatorname{cn} \left[ \frac{L_{1i} - L_{1ij}}{\lambda_{1ij}} \right. \\
 & \left. + F \left[ \arccos \left( \pm \frac{H_{1ij} \lambda_{1ij}}{k_{1ij}} \right), k_{1ij} \right], k_{1ij} \right] = 0, \tag{37}
 \end{aligned}$$

where  $i = 1, 2, j = 1, 2, 3$ . Choosing  $x_{0,1i}^{(\pm)} = \frac{1}{2}(L_{1i} + L_1)$  for case I we have

$$\begin{aligned}
 & \lambda_1 = \lambda_{11} + \lambda_{12}, \lambda_{1i} = \sum_{j=1}^3 \lambda_{1ij}, \\
 & \frac{L_1}{\lambda_1} = 2mK(k_1), \frac{L_{1ij} - L_{1i}}{\lambda_{1ij}} = 2mK(k_{1ij}), \\
 & \frac{L_{1i} - L_1}{2\lambda_{1i}} = 2mK(k_{1i}) - F \left[ \arccos \left( \pm \frac{H_1 \lambda_1}{k_{1i}} \right), k_{1i} \right], \\
 & H_1 \lambda_1 = H_{1ij} \lambda_{1ij} = H > 0, \\
 & n_1 = n_{1i} = n_{1ij} = n, n \in \mathbf{Z} \tag{38}
 \end{aligned}$$

where  $i = 1, 2, j = 1, 2, 3, m \in \mathbf{Z} \setminus \{0\}$ . Assuming the constraints given by Eq. (38) we get

$$k_b = k, g^{(\pm)}(k) \equiv (-1)^{m+1} \arcsin \left\{ k \sqrt{1 - \left(\frac{H}{k}\right)^2} \right\} = 0,$$

and  $k = \pm H$ .

### 4. Stability of solutions

Here we briefly analyse the stability of the obtained solutions using the same method as in [31,32]. We present the computation below for a metric star graph. However, extending the method to tree graphs and other graph topologies is rather trivial.

First, we define the Gibbs free-energy functional on the star graph presented in Fig. 1

$$\Omega_G = \sum_{j=1}^3 \Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H_j \right], \tag{39}$$

with the Gibbs free energy on each bond given by

$$\begin{aligned} \Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H_j \right] &= 2H_j^2 L_j - 2H_j [\phi_j(L_j) - \phi_j(0)] \\ &+ \int_0^{L_j} \left[ 1 - \cos\phi_j(x) + \frac{\lambda_j^2}{2} \left[ \frac{d\phi_j(x)}{dx} \right]^2 \right] dx. \end{aligned} \tag{40}$$

$L_j$  is the length of the bond  $j$ . It is easy to see that the condition  $\delta\Omega_G = 0$  leads to the sine-Gordon equation on a star graph given by Eqs. (1)–(6).

The key role in the stability analysis is played by the second variation of the Gibbs functional given by

$$\delta^2\Omega_G = \sum_{j=1}^3 \int_0^{L_j} \left[ \cos\bar{\phi}_j (\delta\phi_j)^2 + \lambda_j^2 \left( \frac{d\delta\phi_j}{dx} \right)^2 \right] dx.$$

If for the tested solution of the sine-Gordon equation,  $\phi_j(x) = \bar{\phi}_j(x)$

$$\delta^2\Omega_G \left[ \phi_j, \frac{d\phi_j}{dx} \right]_{\phi_j(x)=\bar{\phi}_j(x)} > 0,$$

the solution will be inside the stability region [31,32]. For  $\delta^2\Omega_G \left[ \phi_j, \frac{d\phi_j}{dx} \right]_{\phi_j(x)=\bar{\phi}_j(x)}$  having no definite sign, the solution will be unstable [31,32]. The condition

$$\delta^2\Omega_G \left[ \phi_j, \frac{d\phi_j}{dx} \right]_{\phi_j(x)=\bar{\phi}_j(x)} \geq 0,$$

defines the border of stability (bifurcation point). Furthermore, following [31,32], these conditions can be reformulated in terms of  $\mu_0$ , the lowest eigenvalue of the Sturm–Liouville eigenvalue problem

$$-\lambda_j^2 \frac{d^2\psi_j}{dx^2} + \cos\bar{\phi}_j \psi_j = \mu \psi_j, \tag{41}$$

$$\lambda_1 \frac{d\psi_1}{dx} \Big|_{x=0} = \lambda_2 \frac{d\psi_2}{dx} \Big|_{x=0} = \lambda_3 \frac{d\psi_3}{dx} \Big|_{x=0}, \tag{42}$$

$$\lambda_1 \psi_1|_{x=0} + \lambda_2 \psi_2|_{x=0} + \lambda_3 \psi_3|_{x=0} = 0, \tag{43}$$

$$\frac{d\psi_j}{dx} \Big|_{x=L_j} = 0, \quad j = 1, 2, 3. \tag{44}$$

If the lowest eigenvalue  $\mu = \mu_0$  of this Sturm–Liouville problem is negative, i.e.  $\mu_0 < 0$ , the solution  $\phi = \phi(y)$  corresponds to a saddle point of Eq. (40) and therefore is unstable. The stable solutions minimize the functional  $\Omega_G$  and are characterized by  $\mu_0 > 0$ . The boundary between stable and unstable solutions is determined by the condition  $\mu = 0$ . By solving numerically the problem (41)–(44) we found that  $\mu_0 < 0$  for both cases of the type 1 solutions. For the case I of the solution of type 2 we have  $\mu_0 > 0$ , while for the case II of the solution of type 2 we found that  $\mu_0 < 0$ . Therefore

only case I of the solution of type 2 is stable, while the other solutions are unstable.

The stability analysis for the vertex boundary conditions given by Eqs. (4) and (5) is similar and leads to the same results, i.e. only case I of the type 2 solutions is stable, while the other solutions are unstable.

The above stability analysis is in general applicable also to tree graphs.

### 5. Conclusions

In this paper, we have studied the stationary sine-Gordon equation on simple metric graphs by imposing two different vertex boundary conditions. Exact analytical solutions are obtained for a metric star graph. The constraints that allow exact solutions, have been determined in terms of bond nonlinearity coefficients.

The procedure has been extended to metric tree graphs where explicit solutions are also derived. The stability of the obtained solutions has been analysed. The obtained results can be directly applied to the study of static solitons in 2D branched Josephson junctions in a constant magnetic field, i.e. in T-, Y- and tree-shaped versions of the model studied in [31]. Finally, we note that the method can be extended to the case of “current carrying” boundary conditions studied in Ref. [32]. Note that similar computations can be applied to obtain analytical solutions of the static sine-Gordon equation with different sets of vertex boundary conditions.

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### Appendix A

Here we provide derivation of the vertex boundary conditions given by Eqs. (2)–(3). Starting point for the derivation is the Gibbs free-energy functional given by [31,32]

$$\Omega_G = \sum_{j=1}^3 \Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H_j \right], \tag{45}$$

where Gibbs free energy on each bond is given as

$$\begin{aligned} \Omega_G^{(j)} \left[ \phi_j, \frac{d\phi_j}{dx}; H_j \right] &= 2H_j^2 L_j - 2H_j [\phi_j(L_j) - \phi_j(0)] \\ &+ \int_0^{L_j} \left[ 1 - \cos\phi_j(x) + \frac{\lambda_j^2}{2} \left[ \frac{d\phi_j(x)}{dx} \right]^2 \right] dx. \end{aligned} \tag{46}$$

$$\delta\Omega_G = \sum_{j=1}^3 \int_0^{L_j} \left[ \sin\phi_j(x) \delta\phi_j(x) + \lambda_j^2 \frac{d\phi_j(x)}{dx} \frac{d\delta\phi_j(x)}{dx} \right] dx. \tag{47}$$

Integrating the second term on the right hand side of (47) we have

$$\begin{aligned} \delta\Omega_G &= \sum_{j=1}^3 \int_0^{L_j} \left[ \sin\phi_j(x) - \lambda_j^2 \frac{d^2\phi_j(x)}{dx^2} \right] \delta\phi_j(x) dx \\ &+ \sum_{j=0}^3 \lambda_j^2 \frac{d\phi_j(x)}{dx} \delta\phi_j(x) \Big|_{x=0}^{x=L_j}. \end{aligned} \tag{48}$$

Furthermore, we require [31,32]

$$\delta\Omega_G = 0. \quad (49)$$

This can be fulfilled, in particular, when each sum in Eq. (48) becomes zero.

For the first sum this leads to the stationary sine-Gordon equation (1):

$$\frac{d^2\phi_j(x)}{dx^2} = \frac{1}{\lambda_j^2} \sin\phi_j(x), \quad (50)$$

while, for second term we have

$$\sum_{j=0}^3 \lambda_j^2 \frac{d\phi_j(x)}{dx} \delta\phi_j(x) \Big|_{x=0}^{x=L_j} = 0. \quad (51)$$

Eq. (51) leads to the following vertex boundary conditions:

$$\lambda_1 \frac{d\phi_1}{dx} \Big|_{x=0} = \lambda_2 \frac{d\phi_2}{dx} \Big|_{x=0} = \lambda_3 \frac{d\phi_3}{dx} \Big|_{x=0}, \quad (52)$$

$$\lambda_1 \phi_1|_{x=0} + \lambda_2 \phi_2|_{x=0} + \lambda_3 \phi_3|_{x=0} = 0, \quad (53)$$

$$\frac{d\phi_j}{dx} \Big|_{x=L_j} = 2H_j, \quad j = 1, 2, 3. \quad (54)$$

Similarly, from Eq. (51) one can obtain boundary conditions (4) and (5).

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