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Homoclinic chaos in a pair of parametrically-driven coupled SQUIDs

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Abstract. An rf superconducting quantum interference device (SQUID) consists of a superconducting ring interrupted by a Josephson junction (JJ). When driven by an alternating magnetic field, the induced supercurrents around the ring are determined by the JJ through the celebrated Josephson relations. This system exhibits rich nonlinear behavior, including chaotic effects. We study the dynamics of a pair of parametrically-driven coupled SQUIDs arranged in series. We take advantage of the weak damping that characterizes these systems to perform a multiple-scales analysis and obtain amplitude equations, describing the slow dynamics of the system. This picture allows us to expose the existence of homoclinic orbits in the dynamics of the integrable part of the slow equations of motion. Using high-dimensional Melnikov theory, we are able to obtain explicit parameter values for which these orbits persist in the full system, consisting of both Hamiltonian and non-Hamiltonian perturbations, to form so-called Silnikov orbits, indicating a loss of integrability and the existence of chaos.

1. Introduction

We consider a series of electrical circuits in a line and we could derive the following nonlinear lattice equation [1],

$$\ddot{f}_n + \gamma \dot{f}_n + f_n + \hat{\beta}\sin(2\pi f_n) - \lambda(f_{n+1} - 2f_n + f_{n-1}) = 0 \tag{1.1}$$

The coefficient of the nonlinearity, which corresponds to the so-called Josephson critical current, is modulated in time, which can be realized experimentally by modulating the surrounding temperature. As the temperature also influences the damping due to the unpaired electron, additionally we also consider time-periodically modulated dissipation. Compared to [5], we consider softening nonlinearity as opposed to stiffening one, i.e. the nonlinearity has different sign.

Here, we wish to study the possibility of homoclinic chaos near resonances in typical SQUID lattice, which is described by eq. (2.1). In particular, we consider a specific dimer for the lattice equation (2.1) and show that the unperturbed system has a homoclinic orbit in their collective dynamics. Applying the geomertical method of singular perturbation theory near the resonaces and Melnikov theory of near integrable Hamiltonian system to predict the chaotic behavior near resonances [3]. Moreover, we state the theorem for the exisence of multi-homoclinic orbits near resonance following the references [3, 2]. In Section 2, we perform a multi-scale analysis for the dimer case and obtain amplitude equations to describe the slow dynamics of the system of dimer SQUID. It is in the slow dynamics that the homoclinic orbits are found. In Section 3, we perform the analytical method for the existence of homoclinic orbits in the perturbed system. In particular we calculate the Energy-difference function using the Melnikov integral evaluated on the homoclinic solutions and applying the singular perturbation theory we study the dynamics near resonances. In Section 4, we illustrate numerically our analytical results for the SQUID model.

2. Normal mode amplitude equations

We consider the 1D case of eq (6) in [1]

$$\ddot{f}_n + \gamma\dot{f}_n + f_n + \hat{\beta}\sin(2\pi f_n) - \lambda(f_{n+1} - 2f_n + f_{n-1}) = 0 \tag{2.1}$$

for n=1,2 and $f_0 = f_3 = 0$ we obtain

$$\begin{aligned} \ddot{f}_1 + \gamma\dot{f}_1 + f_1 + \hat{\beta}\sin(2\pi f_1) - \lambda f_2 &= 0 \\ \ddot{f}_2 + \gamma\dot{f}_2 + f_2 + \hat{\beta}\sin(2\pi f_2) - \lambda f_1 &= 0 \end{aligned} \tag{2.2}$$

We could consider a small parameter $\gamma_0 \to \epsilon\hat{\gamma}_0$, $\hat{\beta} = \beta + f_p\cos(\omega_p t)$, $\gamma = \gamma_0$ and drive amplitude $f_p = \epsilon h$. The normal mode frequencies for the linear system are:

$$\omega_1^2 = 1 + 2\pi\beta - \lambda, \omega_2^2 = 1 + 2\pi\beta + \lambda \tag{2.3}$$

We use multiple time scales to express the displacements:

$$f_{1,2}(t) = \frac{\sqrt{3\epsilon}}{2}(A_1(T)\mathrm{e}^{i\omega_1 t} \pm A_2(T)\mathrm{e}^{i\omega_2 t} + c.c) + \epsilon^{3/2}f_{1,2}^{(1)}(t) + \dots \tag{2.4}$$

where f_1 is taken with the positive sign and f_2 with the negative sign, slow time $T = \epsilon t$. The drive frequency ω_p is related with ω_2: $\omega_p = 2\omega_2 + \epsilon\Omega$, $\omega_1 = \omega_2 + 2\epsilon\Omega_1$. Substituting multiple time scale expression to the system of f_1, f_2 generates secular terms that yield two coupled equations for the complex amplitudes $A_{1,2}$. Express the complex amplitudes using real amplitudes and phases as

$$A_1(T) = a_1(T)\mathrm{e}^{i(x_1(T)+(\Omega/2-2\Omega_1)T)}, \quad A_2(T) = a_2(T)\mathrm{e}^{i(x_2(T)+\Omega T/2)} \tag{2.5}$$

the real and imaginary parts of the two secular amplitude equations become:

$$\begin{aligned} \frac{\mathrm{d}a_1}{\mathrm{d}T} &= \frac{3\pi^3\beta}{2\omega_1}a_1a_2^2\sin(2(x_2 - x_1) + \frac{\pi\hat{h}}{2\omega_1}\sin(2x_1)a_1 - \frac{1}{2}a_1\hat{\gamma}_0 \\ \frac{\mathrm{d}x_1}{\mathrm{d}T} &= -\frac{1}{2}\Omega + 2\Omega_1 - \frac{3\pi^3\beta}{2\omega_1}(a_1^2 + 2a_2^2 + a_2^2\cos2(x_2 - x_1)) + \frac{\pi\hat{h}}{2\omega_1}\cos(2x_1) \\ \frac{\mathrm{d}a_2}{\mathrm{d}T} &= \frac{3\pi^3\beta}{2\omega_1}a_2a_1^2\sin(2(x_1 - x_2) + \frac{\pi\hat{h}}{2\omega_2}\sin2x_2a_2 - \frac{1}{2}a_2\hat{\gamma}_0 \\ \frac{\mathrm{d}x_2}{\mathrm{d}T} &= -\frac{\Omega}{2} - \frac{3\pi^3\beta}{2\omega_2}(a_2^2 + 2a_1^2 + a_1^2\cos2(x_1 - x_2)) + \frac{\pi\hat{h}}{2\omega_2}\cos(2x_2) \end{aligned} \tag{2.6}$$

2.1. Analytical expressions for homoclinic orbits

We performed a multi-scale analysis [4] for the dimer case and obtained amplitude equations to describe the slow dynamics of the system of dimer SQUID. It is in the slow dynamics that the homoclinic orbits are found. The homoclinic orbits given by

$$B^h(T,I) = \frac{2\delta a^2}{q\cosh(2aT)+p} \tag{2.7}$$

$$\theta^h(T,I) = \tan^{-1}\Big(\sqrt{\frac{I(\delta-3)+\Omega_1}{I(1-\delta)-\Omega_1}}\tanh(aT)\Big) \tag{2.8}$$

$$\chi_1^h(T,I) = \frac{-a(\delta^2-1)}{\sqrt{p^2-q^2}}\tan^{-1}[-\sqrt{\frac{p-q}{p+q}}\tanh(aT)] + (\delta I - \frac{\Omega}{4})T + x_1(0) \tag{2.9}$$

$$\phi^h(T,I) = \chi_1^h(T,I) - \theta^h(T,I) \tag{2.10}$$

where $q = I(\delta^2-1)+2\delta\Omega_1$, $p = -\Omega_1(\delta^2-4\delta+1) - I(\delta^3-6\delta^2+7\delta-2)$ and $a^2 = \Omega_1^2 - 2I\Omega_1(\delta-2) - I^2(\delta-3)(\delta-1)$.

The orbits given by Eqs.2.7-2.10 are homoclinic to Π_0. In Fig. 2 we can see some of them. At resonance, for $I = I_0 = -\Omega/4\delta$, the orbits are heteroclinic, connecting fixed points that are $\Delta\phi$ apart, where $\Delta\phi = \Delta x_1 - \Delta\theta$. Therefore

$$\Delta\phi = \frac{2a(\delta^2-1)}{\sqrt{p^2-q^2}}\tanh^{-1}[\sqrt{\frac{p-q}{p+q}}] - 2\tan^{-1}\sqrt{\frac{I_0(\delta-3)+\Omega_1}{I_0(1-\delta)-\Omega_1}}$$

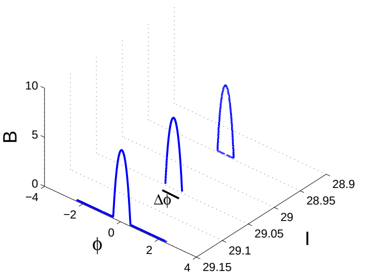

Figure 1. Orbits homoclinic to $\Pi_0 = \big\{(x,y,I,\phi) : x = y = 0, 1 < \delta < 3, I > \frac{\Omega_1}{3-\delta}\big\}$. For $I = I_0$ (the orbit in the middle),$d\phi/dT = 0$ on Π_0, and the orbit is heteroclinic, connecting fixed points on Π_0 that are $\Delta\phi$ apart. For $I \lesseqgtr I_0, d\phi/dT \lesseqgtr 0$ on Π_0. The parameters are $\delta = 1.49704$, $\Omega = -110$, $\Omega_1 = 27.6085$.

3. Homoclinic intersections in the perturbed system

In this section we calculate the Energy-difference function using the Melnikov integral evaluated on the homoclinic solutions. We compute that the energy-difference function is

$$\begin{aligned}\Delta^N\mathcal{H}(\phi_0) \;=\; & \cos(2(\phi_0+N\Delta\phi)) - \cos(2\phi_0) + \\ & \frac{\sin\frac{(N\Delta\phi)}{2}}{\sin\frac{\Delta\phi}{2}}2(\sin 2\phi_0)\sin(N\Delta\phi) - \\ & \frac{2\xi N}{h\delta}(-2\Delta\phi - \Delta\theta - \frac{\tilde{L}}{I_0})\end{aligned} \tag{3.1}$$

When the dissipation parameter $\xi < \frac{\sin N\Delta\phi(1+2\frac{\sin N\Delta\phi/2}{\sin\Delta\phi/2})}{\frac{2N}{h\delta}(-2\Delta\phi-\Delta\theta-\frac{\tilde{L}}{I_0})}$ then the energy function admits transverse zeros.

3.1. Dynamics near resonance

Now applying the singular perturbation theory we study the dynamics near resonances. The equations that describe the dynamics on $M_\varepsilon = \big\{(x,y,I,\phi) : x = x^\varepsilon(I,\phi), y = y^\varepsilon(I,\phi)\big\}$ near the resonace $I = I_0$ are

$$\frac{dI}{dT} = \varepsilon hI\delta\sin(2\phi) - \varepsilon\gamma_0 I, \frac{d\phi}{dT} = -\delta I - \frac{\Omega}{4} + \frac{\varepsilon h\delta}{2}\cos(2\phi) \tag{3.2}$$

According to [4] and [3] in order to study the slow dynamics, which is induced by the perturbation on M_ε near resonance, we set a slow variable $I = I_0 + \sqrt{\varepsilon}J$ into Eqs. (3.2) along with a slow time scale $\tau = \sqrt{\varepsilon}T$, and we obtain the Hamiltonian system in (J, ϕ)

$$H(J,\phi) = -\frac{\delta J^2}{2} + I_0\gamma_0\phi + \frac{1}{2}I_0 h\delta\cos(2\phi) \tag{3.3}$$

3.2. A homoclinic connection to the sink p_ε

We are in a position to show the existence of an orbit homoclinic to the sink p_ε. Here we illustrate numerically our analytical results for the SQUID model. In Fig.2a we show that the unperturbed heteroclinic orbit, which asympotes to p_c as $T \to \infty$, returns back to a point on the circle of fixed points that is inside the homoclinic separatrix loop. In order to have Silnikov Chaos [4] the condition

$$\phi_s < \phi_c + \Delta\phi < \phi_m \tag{3.4}$$

has to be satisfied. The ϕ values of the fixed points are shown in Fig.2b along with $\phi_c + \Delta\phi$ and ϕ_m for a particular value of Ω. The parameter values for which these ϕ values satisfy the condition 3.4 are displayed in Fig.2c.

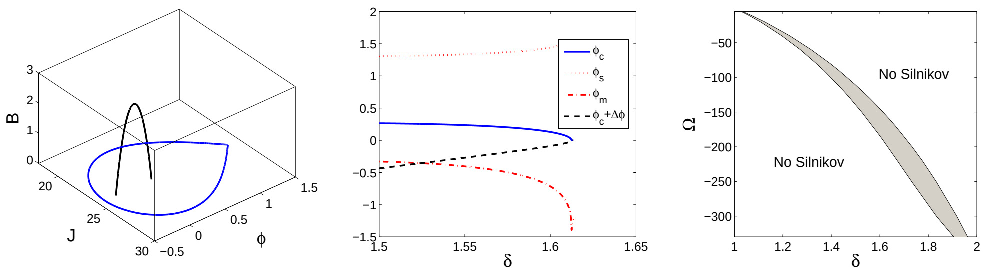

Figure 2. a) The heteroclinic orbit given by Eqs. 2.7 and 2.10 with $I = I_0$, superimposed with the phase portrait of the unperturbed scaled system on Π_ε near resonance. The parameters are $\delta = 1.55, \Omega = -150, \Omega_1 = 35, \zeta = 0.45, h = 1$ ($\xi = \gamma/h$ and $\zeta = \xi/\delta$). This value of ζ sets $\phi_c = 0.233383$ and we can see from the figure that $\phi_s < \phi_c + \Delta\phi < \phi_m$. b) The values of $\phi_s, \phi_c, \phi_c + \Delta\phi$ and ϕ_m as functions of δ , for $\Omega = -150$. For $\delta > 1.524$ the condition is satisfied. c) In gray are indicated the parameter values for which the condition 3.4 is satisfied. In figures b and c: $\varepsilon = 0.01$

Summary: In this work we studied chaotic dynamics of a pair of parametrically-driven coupled SQUIDs arranged in series and we found under which conditions it can exist by using high dimentional Melnikov theory [4].

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References

[1] N.Lazarides, G.P. Tsironis and N. Eleftheriou, *Nonlinear Phenomena in Complex Systems*,11,250, 2008.
[2] G. Mennon, G. Haller and V. Rothos, *Phys Lett A*,263,175, 1999.
[3] G. Haller, *Chaos near Resonances*, Springer, New York, 1999.
[4] G. Kovacic and S. Wiggins, *Physica D*, 57, 185, 1992; G. Kovacic, *SIAM J Math Anal*, 26, 1611, 1995.
[5] Eyal Kenig, Yuriy A. Tsarin, Ron Lifshitz, *Phys. Rev. E* 84 (2011) 016212.