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NONLOCAL REDUCTIONS OF THE ABLOWITZ–LADIK EQUATION

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Our purpose is to develop the inverse scattering transform for the nonlocal semidiscrete nonlinear Schrödinger equation (called the Ablowitz–Ladik equation) with $\mathcal{PT}$ symmetry. This includes the eigenfunctions (Jost solutions) of the associated Lax pair, the scattering data, and the fundamental analytic solutions. In addition, we study the spectral properties of the associated discrete Lax operator. Based on the formulated (additive) Riemann–Hilbert problem, we derive the one- and two-soliton solutions for the nonlocal Ablowitz–Ladik equation. Finally, we prove the completeness relation for the associated Jost solutions. Based on this, we derive the expansion formula over the complete set of Jost solutions. This allows interpreting the inverse scattering transform as a generalized Fourier transform.

Keywords: integrable system, soliton, PT symmetry, nonlocal reduction, Riemann–Hilbert problem

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1. Introduction

Completely integrable infinite-dimensional systems are a subject of constant interest and many investigations in different areas of mathematics and physics over almost the last five decades [1]–[3] and appear in a wide range of applications from differential geometry to classical and quantum field theory, fluid mechanics, and optics.

A special class of completely integrable infinite-dimensional systems is the class of PDEs integrable by the inverse scattering method (ISM) [1], [3]. The nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + 2|q^2|q = 0, \qquad q = q(x,t), \tag{1}$$

appeared in a very early stage of the development of the ISM [1], [3], [4] as one of the classical examples of integrable equations by the ISM and has attracted significant attention of the scientific community [5]–[7]. It appears as an universal model for weakly nonlinear dispersive waves, nonlinear optics, and plasma physics [8].

The NLS model has been generalized in several directions. The first is to consider multicomponent generalizations. The first multicomponent/vector generalization of (1) was proposed by Manakov in 1974 (see [3])

$$i\mathbf{v}_t + \mathbf{v}_{xx} + 2(\mathbf{v}^\dagger, \mathbf{v})\mathbf{v} = 0, \qquad \mathbf{v} = \mathbf{v}(x,t), \tag{2}$$

where $\mathbf{v}$ is an n-component complex-valued vector and $(\,\cdot\,,\,\cdot\,)$ is the standard scalar product. It is again integrable by the ISM [1]–[3], [8]. The two component VNLS equation (called the Manakov model) appears

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in studies of electromagnetic waves in optical media. Another direction, motivated by applications of differential geometric and Lie algebraic methods to soliton-type equations [9]–[18] (see, e.g., [2] for a detailed review) led to the discovery of a close relation between the multicomponent (matrix) NLS equations and homogeneous and symmetric spaces [15].

The first integrable discretization of NLS equation (1) was proposed by Ablowitz and Ladik and has the form [19]–[21]

$$iQ_{n,t} = \frac{1}{h^2}(Q_{n+1} - 2Q_{n+1} + Q_{n-1}) - \epsilon |Q_n|^2 (Q_{n+1} + Q_{n-1}), \qquad \epsilon = \pm 1. \tag{3}$$

It is a differential–difference or semidiscrete equation (discrete in space and continuous in time) and is in fact a $O(h^2)$ finite-difference approximation of (1). The corresponding scattering problem is usually called the Ablowitz–Ladik (AL) scattering problem [8], [22]–[28]. Equation (3) also has several physical applications: it describes the dynamics of anharmonic lattices [29], self-trapping on a dimer [30], various types of Heisenberg spin chains [31], [32], and so on. Various discretizations of the NLS models were later studied [5], [33]–[38] including perturbation effects [39], [40].

A nonlocal integrable equation of the NLS type with $\mathcal{PT}$ symmetry, due to the invariance of the so-called self-induced potential $V(x,t) = \psi(x,t)\psi^*(-x,-t)$ under the combined action of parity and time reversal symmetry, was recently proposed in [41], [42]. There, a one-soliton solution was derived for this model, and it was shown that it develops singularities in a finite time. Soon after, nonlocal $\mathcal{PT}$-symmetric generalizations were found for the AL model in [43]. All these models are integrable by the ISM [44].

Nonlocal reductions of NLS equation (1) and AL equation (3) are particularly interesting for applications in $\mathcal{PT}$-symmetric optics, especially in developing a theory of electromagnetic waves in artificial heterogeneous media [45], [46] (see, e.g., [47], [48] for an up-to-date review).

The initial interest in such systems was motivated by quantum mechanics [49], [50]. It was shown in [49] that quantum systems with a non-Hermitian Hamiltonian admit states with real eigenvalues, i.e., the Hermiticity of the Hamiltonian is not a necessary condition for a real spectrum. New quantum mechanics can be developed using such Hamiltonians [49]–[52]. The starting point is the fact that in the case of a non-Hermitian Hamiltonian with a real spectrum, the modulus of the wave function for the eigenstates is independent of time even in the case of complex potentials.

The first pseudo-Hermitian Hamiltonian with a real spectrum historically was the $\mathcal{PT}$-symmetric Hamiltonian in [49], [53], [54]. Pseudo-Hermiticity here means that the Hamiltonian $\mathcal{H}$ commutes with the operators of spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$: $\mathcal{PTH} = \mathcal{HPT}$. The actions of these operators is defined as $\mathcal{P}$: $x \to -x$ and $\mathcal{T}$: $t \to -t$. If we assume that the wave function is a scalar, then this leads to the action of the operator of spatial reflection on the space of states $\mathcal{P}\psi(x,t) = \psi(-x,t)$ and $\mathcal{T}\psi(x,t) = \psi^*(x,-t)$. As a result, the Hamiltonian and the wave function are $\mathcal{PT}$-symmetric if $\mathcal{H}(x,t) = \mathcal{H}^*(-x,-t)$ and $\psi(x,t) = \psi^*(-x,-t)$. Here, we also used the fact that the parity operator $\mathcal{P}$ is linear and unitary while the time-reversal operator $\mathcal{T}$ is antilinear and antiunitary.

The action of the operators $\mathcal{P}$ and $\mathcal{T}$ on the Hamiltonian induces an action on the associated scattering problem (see (6) below) and on its potential (8):

$$\mathcal{P}Q_n(t) = Q_{-n}(t), \qquad \mathcal{T}Q_n(t) = Q_n^*(-t).$$

This leads to the reduction (symmetry) condition

$$Q_n^-(t) = \pm (Q^+)^*_{-n}(t). \tag{4}$$

As a result, we obtain the nonlocal AL equation with $\mathcal{PT}$ symmetry, proposed in [43]:

$$iQ^+_{n,\tau} = (Q^+_{n+1} - 2Q^+_n + Q^+_{n-1}) - \epsilon Q^+_n (Q^+)^*_{-n} (Q^+_{n+1} + Q^+_{n-1}). \tag{5}$$

Our purpose here is to develop the inverse scattering transform for (5), to study the spectral properties of the associated Lax operators (6) and (7), and to derive one- and two-soliton solutions.

This paper is organized as follows. In Sec. 2, we briefly outline the structure of the semidiscrete Lax representation, the corresponding semidiscrete (differential–difference) zero-curvature equation, and the resulting differential–difference equations. In Sec. 3, we present the direct scattering transform for the nonlocal AL equation. This includes the Jost solutions, the scattering matrix, the scattering data, and the fundamental analytic solutions (FAS). In Sec. 4, we formulate a Riemann–Hilbert problem (RHP, in additive form) for the FAS on the continuous spectrum of the discrete Lax operator. Based on this, we derive the one- and two-soliton solutions of (5). Finally, in Sec. 5, we describe the spectral properties of the discrete Lax operator, prove the completeness relation for the Jost solutions, and derive an expansion formula over the complete set of Jost solutions.

2. Preliminaries

The starting point here is the semidiscrete analogue of the Lax (or zero-curvature) representation. The initial nonlinear evolutionary equation (3) can be represented as a compatibility condition for two linear systems:

$$\Psi_{n+1}(z,t) = L_n(z,t)\Psi_n(z,t), \quad n \in \mathbb{N}, \tag{6}$$

$$\Psi_{n,t}(z,t) = M_n(z,t)\Psi_n(z,t), \tag{7}$$

where

$$L_n(z,t) = \begin{pmatrix} z & Q_n^+(t) \\ Q_n^-(t) & z^{-1} \end{pmatrix} \tag{8}$$

is an element of the Lie group $SL(2,\mathbb{C})$ and $M_n(z,t)$ is an element of the corresponding Lie algebra $sl(2,\mathbb{C})$. Here, we also assume that $Q_n^\pm(t)$ are complex-valued functions satisfying $\sum_{n=-\infty}^{\infty} Q_n^\pm(t) < \infty$. The compatibility condition (i.e., the semidiscrete analogue of the zero-curvature representation) of (6) and (7) becomes

$$M_{n+1} = L_{n,t}L_n^{-1} + L_nM_nL_n^{-1}, \tag{9}$$

where the operator M_n becomes $M_n(z,t) = V_n(z,t) + \Omega(z)$ with

$$V_n(z,t) = i\begin{pmatrix} Q_n^+Q_{n-1}^- & -(z^{-1}Q_{n-1}^+ - zQ_n^+) \\ (z^{-1}Q_n^- - zQ_{n-1}^-) & -Q_n^-Q_{n-1}^+ \end{pmatrix}, \qquad \Omega = -\frac{i}{2}(z-z^{-1})^2\sigma_3. \tag{10}$$

Equation (3) together with appropriate boundary conditions is integrable by the ISM [19]–[21]. In addition, Eq. (3) is one of the members of the integrable hierarchy associated with spectral problem (6).

Discrete compatibility condition (9) yields the system of differential–difference equations (without using the involution)

$$\begin{aligned} iQ_{n,\tau}^+ &= (Q_{n+1}^+ - 2Q_n^+ + Q_{n-1}^+) - Q_n^+Q_n^-(Q_{n+1}^+ + Q_{n-1}^+), \\ -iQ_{n,\tau}^- &= (Q_{n+1}^- - 2Q_n^- + Q_{n-1}^-) - Q_n^-Q_n^+(Q_{n+1}^- + Q_{n-1}^-). \end{aligned} \tag{11}$$

If we impose the standard symmetry condition (involution) $Q_n^-(t) = \epsilon(Q_n^+(t))^*$, then we obtain (3). If we set the nonlocal involution $Q_n^-(t) = \epsilon(Q_{-n}^+(t))^*$, then we obtain (5).

Finally, we note that the AL Lax operator $L_n(z)$ can be transformed into a spectral (eigenvalue) problem $\mathcal{L}_n(z)\Psi_n(z) = 0$, where

$$\mathcal{L}_n(z) = \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix} + U_n - z\mathbb{I}, \qquad U_n = \begin{pmatrix} 0 & Q_n^+ \\ Q_{n-1}^- & 0 \end{pmatrix}. \tag{12}$$

Here, $D_\pm$ are the shift operators $D_\pm\Psi_n(z) = \Psi_{n\pm1}(z)$, and $\mathbb{I}$ is the 2×2 identity matrix [25], [26].

Spectral problem (12) and the generic differential–difference system (8) have a symmetry of the form

$$\mathbf{C}[L_n(z)] := BL_{-n}(z^*)^\dagger B^{-1} = L_n(z), \tag{13}$$

where $\mathbf{C}$ is an automorphism of the Lie group $SL(2,\mathbb{C})$. The particular choice $B = \operatorname{diag}(1,-1)$ of the realization of $\mathbf{C}$ gives $Q_n^-(t) = \epsilon(Q_n^+(t))^*$. Therefore, the potentials U_n in (12) and V_n in (10) reduce to

$$U_n = \begin{pmatrix} 0 & Q_n^+ \\ \epsilon(Q_{n-1}^+(t))^* & 0 \end{pmatrix},$$

$$V_n(z,t) = i\begin{pmatrix} -\epsilon Q_n^+(Q_{1-n}^+)^* & -(z^{-1}Q_{n-1}^+ + zQ_n^+) \\ \epsilon(z^{-1}(Q_{-n}^+)^* - z(Q_{1-n}^+)^*) & \epsilon(Q_{-n}^+)^* Q_{n-1}^+ \end{pmatrix}.$$

As a result, generic system (11) reduces to (5).

3. Direct scattering transform

3.1. Jost solutions and scattering data. The eigenfunctions of $L_n(z)$ and $M_n(z)$ of (6) and (7) are defined by their asymptotic forms (the so-called Jost solutions) as $|n| \to \infty$ (see, e.g., [1]–[3]):

$$\begin{aligned} \psi_n(z) &\to \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } n \to +\infty, \\ \phi_n(z) &\to \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } n \to -\infty. \end{aligned} \tag{14}$$

Along with the standard Jost solutions, we can define "renormalized" Jost solutions from the functions $\psi_n(z)$ and $\phi_n(z)$ that satisfy scattering problem (6),

$$\xi_n(z) = \psi_n(z)\mathbf{Z}^{-n}, \qquad \varphi_n(z) = \phi_n(z)\mathbf{Z}^{-n}, \tag{15}$$

where $\mathbf{Z} = \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$ and the eigenfunctions ξ_n and φ_n are solutions of the respective difference equations

$$\xi_{n+1} = (\mathbf{Z} + \widetilde{\mathbf{Q}}_n)\xi_n\mathbf{Z}^{-1}, \qquad \varphi_{n+1} = (\mathbf{Z} + \widetilde{\mathbf{Q}}_n)\varphi_n\mathbf{Z}^{-1}, \tag{16}$$

where $\widetilde{\mathbf{Q}}_n = \left(\begin{smallmatrix} 0 & Q_n^+ \\ Q_n^- & 0 \end{smallmatrix}\right)$, with the canonical boundary conditions

$$\lim_{n\to\infty} \xi_n(z) = \mathbb{I}, \qquad \lim_{n\to-\infty} \varphi_n(z) = \mathbb{I}. \tag{17}$$

The two Jost solutions $\phi_n(z)$ and $\psi_n(z)$ are related by the scattering matrix:

$$\phi_n(z) = \psi_n(z)T(z), \qquad T(z) = \begin{pmatrix} a^+(z) & -b^-(z) \\ b^+(z) & a^-(z) \end{pmatrix}. \tag{18}$$

Nonlocal involution (13) imposes a symmetry condition as Jost solutions associated with (8) and become

$$\mathbf{C}(\psi_{-n}^\dagger((z)^*,t)) = B\psi_{-n}^\dagger((z)^*,t)B^{-1} = \phi_n(z,t). \tag{19}$$

3.2. Fundamental analytic solutions. Important tools for reducing the inverse scattering problem to a RHP are the FAS $\chi^{\pm}(x,t,\lambda)$. Their construction is based on the Gauss decomposition of $T(\lambda,t)$ (see [3], [55], [56])

$$\begin{aligned} \chi_n^+(z) &= \psi_n(z)T^-(z) = \phi_n(z)S^+(z), \\ \chi_n^-(z) &= \psi_n(z)T^+(z) = \phi_n(z)S^-(z), \end{aligned} \tag{20}$$

where

$$\begin{aligned} T^-(z) &= \begin{pmatrix} a^+(z) & 0 \\ b^+(z) & 1 \end{pmatrix}, \qquad T^+(z) = \begin{pmatrix} 1 & -b^-(z) \\ 0 & a^-(z) \end{pmatrix}, \\ S^+(z) &= \begin{pmatrix} 1 & \beta^-(z) \\ 0 & \alpha^+(z) \end{pmatrix}, \qquad S^-(z) = \begin{pmatrix} \alpha^-(z) & 0 \\ -\beta^+(z) & 1 \end{pmatrix} \end{aligned} \tag{21}$$

are the factors in the Gauss decomposition of the associated scattering matrix $T(z)$,

$$T(z) = T^-(z)\widehat{S}^+(z) = T^+(z)\widehat{S}^-(z).$$

They are expressed in terms of the matrix elements of the scattering matrix $T(z)$ and its inverse

$$\widehat{T}(z) = \begin{pmatrix} \alpha^-(z) & \beta^-(z) \\ -\beta^+(z) & \alpha^+(z) \end{pmatrix}. \tag{22}$$

This construction ensures that $\xi^{\pm}(z)$ are analytic functions of z for $z \in \Omega_{\pm}$.

On the unit circle $|z| = 1$ (i.e., on the continuous spectrum of $L_n(z)$), the two FAS are linearly dependent:

$$\widetilde{\chi}_n^+(z) - \widetilde{\chi}_n^-(z) = \widetilde{\chi}_n^-(z)\mathbf{G}_n(z), \qquad |z| = 1, \tag{23}$$

where the sewing function $\mathbf{G}_n(z,t)$ can be expressed in terms of $\rho^{\pm}(t,z)$,

$$\mathbf{G}_n(z,t) = \begin{pmatrix} \rho^+\rho^- & z^{2n}\rho^- \\ z^{-2n}\rho^+ & 0 \end{pmatrix}, \qquad \widetilde{\chi}_n^-(z) \to \mathbb{I} \quad \text{as } |z| \to \infty. \tag{24}$$

The independent matrix elements of $\mathbf{G}_n(z,t)$ together with the discrete spectrum of $L_n(z)$ form the minimal set of scattering data for L_n.

Nonlocal involution (13) imposes a symmetry condition on the FAS and the scattering matrix:

$$\mathbf{C}(\chi_{-n}^{\{-,\dagger\}}((z)^*,t)) = B\chi_{-n}^{\{-,\dagger\}}((z)^*,t)B^{-1} = \chi_n^+(z,t) \tag{25}$$

and

$$\mathbf{C}(T^{\dagger}((z)^*,t)) = BT^{\dagger}((z)^*,t)B^{-1}. \tag{26}$$

As a result, we obtain

$$a^{\pm}(z,t) = (a^{\pm}(z^*,t))^*, \qquad b^{\pm}(z,t) = (b^{\mp}(z^*,t))^*. \tag{27}$$

3.3. Asymptotic behavior of the FAS. Until the end of this section, we assume that involution (13) holds.

3.3.1. Asymptotic behavior of the FAS for $|z| = 1$. Discrete scattering problem (6) can have discrete eigenvalues. This can occur if $a^{\pm}(z_j) = 0$ for some z_j. We here assume that this cannot happen on the continuous spectrum. On the zeroes z_j of $a^{\pm}(z)$, the two Jost solutions become proportional:

$$\varphi_n^{\pm}(z_j) = \pm b_j^{\pm} z_j^{\mp 2n} \xi_n^{\pm}(z_j). \tag{28}$$

We assume that $a^+(z)$ has S simple zeros $\{z_j \colon |z_j| > 1\}_{j=1}^S$ and $a^-(z)$ has S simple zeros $\{z_j \colon |z_j| < 1\}_{j=1}^S$, i.e., the number of zeros inside the unit circle is equal to the number of zeros outside the unit circle. By Eqs. (28), we then have

$$\operatorname{Res}(\widetilde{\varphi}_n^{\pm}, z_j^{\pm}) = \frac{\varphi_n^{\pm}(z_j^{\pm})}{\dot{a}^{\pm}(z_j^{\pm})} = \pm \frac{b_j^{\pm}(z_j^{\pm})^{\mp 2n} \xi_n^{\pm}}{\dot{a}^{\pm}(z_j^{\pm})} = \pm (z_j^{\pm})^{\mp 2n} C_j^{\pm} \xi_n^{\pm}(z_j^{\pm}), \tag{29}$$

where $C_j^{\pm}$ denotes the normalization constants.

3.3.2. Asymptotic behavior of the FAS as $|z| \to \infty$ and $|z| \to 0$. The Laurent expansions of the FAS $\chi_n^{\pm}(z)$ then have the forms

$$\chi_n^+(z) = \begin{pmatrix} 1 + O(z^{-2}),\ n \text{ even} & c_n^{-1} z^{-1} Q_n^+ + O(z^{-3}),\ n \text{ odd} \\ z^{-1} Q_{n-1}^- + O(z^{-3}),\ n \text{ odd} & c_n^{-1} + O(z^{-2}),\ n \text{ even} \end{pmatrix}$$

as $|z| \to \infty$ and

$$\chi_n^-(z) = \begin{pmatrix} c_n^{-1} + O(z^2),\ n \text{ even} & z Q_{n-1}^+ + O(z^3),\ n \text{ odd} \\ c_n^{-1} z Q_n^- + O(z^3),\ n \text{ odd} & 1 + O(z^2),\ n \text{ even} \end{pmatrix}$$

as $|z| \to 0$. It follows from the analytic properties of the eigenfunctions of $L_n(z)$ that $a^+(z)$ has an analytic extension in the region $|z| \to \infty$:

$$W(\varphi_n^+, \xi_n^+) = W \begin{pmatrix} 1 + O(z^{-2}),\ n \text{ even} & c_n^{-1} z^{-1} Q_n + O(z^3),\ n \text{ odd} \\ z^{-1} R_{n-1} + O(z^{-3}),\ n \text{ odd} & c_n^{-1} + O(z^{-2}),\ n \text{ even} \end{pmatrix}.$$

Then $\chi_n^+(z)$ is analytic as $|z| \to \infty$,

$$a^+(z) = 1 - O(z^{-2}),\ n \text{ even}, \quad \text{as } |z| \to \infty. \tag{30}$$

Similar arguments apply for finding the Laurent series expansions for $a_n^-(z)$:

$$W(\xi_n^-, \varphi_n^-) = W \begin{pmatrix} c_n^{-1} + O(z^2),\ n \text{ even} & z Q_{n-1} + O(z^3),\ n \text{ odd} \\ c_n^{-1} z R_n + O(z^3),\ n \text{ odd} & 1 + O(z^2),\ n \text{ even} \end{pmatrix}. \tag{31}$$

Hence, $\chi_n^-(z)$ is analytic around $|z| = 0$, and

$$a^-(z) = 1 - O(z^2),\ n \text{ even}, \quad \text{as } |z| \to 0. \tag{32}$$

The scattering coefficients can be written as explicit sums of the eigenfunctions:

$$\begin{aligned} a^+(z) &= 1 + \sum_{k=-\infty}^{+\infty} z^{-1} Q_k^+ \varphi_k^{(2),+}, \qquad & b^+(z) &= \sum_{k=-\infty}^{+\infty} z^{2k-1} Q_k^- \varphi_k^{(1),+}, \\ a^-(z) &= 1 + \sum_{k=-\infty}^{+\infty} z Q_k^- \varphi_k^{(1),-}, \qquad & b^-(z) &= -\sum_{k=-\infty}^{+\infty} z^{-2k+1} Q_k^+ \varphi_k^{(2),-}. \end{aligned} \tag{33}$$

4. The Riemann–Hilbert problem and soliton solutions

It is well known that the inverse scattering transform for the Lax operator $L_n(z)$ can be reduced to a Riemann–Hilbert boundary value problem on the complex plane. The contour where the values of the analytic functions are specified is the continuous spectrum of $L_n(z)$ given by (6); in the case of the AL equation, it is the unit circle $|z| = 1$. If the Lax operator $L_n(z)$ has discrete eigenvalues, then the resulting RHP is singular. Here, we restrict ourself to the so-called balanced RHPs: we consider only problems with an equal number of singularities inside and outside the boundary contour, i.e., we assume that the number of zeros of $a^+(z)$ is equal to the number of zeros of $a^-(z)$. The reflectionless case $b^\pm(z) = 0$ corresponds to soliton solutions of (5) with $\epsilon = -1$.

Symmetries and symmetry reductions. Because the expansions of $a^\pm(z)$ presented in Sec. 3.2 contain only even powers of z^{-1} and z, it follows that if $z_j^\pm$ are zeros of $a^\pm(z)$, then $-z_j^\pm$ are also zeros of $a^\pm(z)$. This implies that

$$C_j^\pm(-z_j^\pm) = \frac{b^\pm(-z_j^+)}{\dot{a}^\pm(-z_j^\pm)} = C^\pm(z_j^+), \tag{34}$$

where $b_j^- = -b_j^+$, $\rho^+(-z) = -\rho^+(z)$, and $\rho^-(-z) = -\rho^-(z)$. Nonlocal involution (13) acts on the reflection coefficient and the normalization constant producing the constraint

$$C_j^-(z_j^-) = \frac{(b^+(z_j^{+,*}))^*}{(\dot{a}^-(z_j^{-,*}))^*}, \qquad \rho^-(z) = \frac{(b^+(z^*))^*}{(a^-(z^*))^*}. \tag{35}$$

The case of poles. If the FAS $\widetilde{\varphi}_n^\pm(z)$ have poles, then the method for solving the RHP requires an extra step involving a contour integration. The starting point is the relations between the eigenfunctions,

$$\widetilde{\varphi}_n^\pm = \frac{\varphi_n^\pm}{a^\pm}(z) = \xi_n^\mp(z) \pm z^{\mp 2n}\rho^\pm(z)\xi_n^\pm(z). \tag{36}$$

We apply the contour integration method to the integral representations

$$\begin{aligned} \mathcal{J}_{1,n}(z) &= \frac{1}{2\pi i}\left(\oint_{\gamma^+} \frac{d\omega\, \varphi_n^+(\omega)}{(\omega - z)a^+(\omega)} - \oint_{\gamma^-} \frac{d\omega\, \xi_n^-(\omega)}{\omega - z}\right), \\ \mathcal{J}_{2,n}(z) &= \frac{1}{2\pi i}\left(\oint_{\gamma^+} \frac{d\omega\, \xi_n^+(\omega)}{\omega - z} - \oint_{\gamma^-} \frac{d\omega\, \varphi_n^-(\omega)}{(\omega - z)a^-(\omega)}\right), \end{aligned} \tag{37}$$

where the integration contours $\gamma_\pm$ are shown in Fig. 1. Here, we present the detailed evaluation of one of the integrals ($\mathcal{J}_{2,n}$). The other can be evaluated similarly.

We recall that (1) $1/a^+(z)$ has a simple pole at $z = z_j^+$, (2) $1/a^-(z)$ has a simple pole at $z = z_j^-$, (3) ξ_n^- and ξ_n^+ have no poles, the integrand of the first integral in $\mathcal{J}_{2,n}(z)$ hence has only a pole at $z = \omega$, and (4) the outside contour is a negatively oriented while the inside contour is a positively oriented. Therefore, for $z \in \Omega_+$, we have

$$\mathcal{J}_{2,n}(z) = \xi_n^+(z) - \sum_{j=1}^{S}\left[\frac{\varphi_n^-(z_j^-)}{(z - z_j^-)\dot{a}_j^-} + \frac{\varphi_n^-(-z_j^-)}{(z + z_j^-)\dot{a}_j^-}\right]. \tag{38}$$

The integral along the unit circle $\Omega = z \in \mathbb{C}$: $|z| = 1$ is equal to

$$\frac{1}{2\pi i}\left(\oint_{\gamma^+} \frac{d\omega\, \xi_n^+(\omega)}{\omega - z} - \oint_{\gamma^-} \frac{d\omega\, \varphi_n^-(\omega)}{(\omega - z)a^-(\omega)}\right) = \frac{1}{2\pi i}\oint_{\Omega} \frac{d\omega}{\omega - z}\omega^{2n}\rho^-(z)\xi_n^-(z). \tag{39}$$

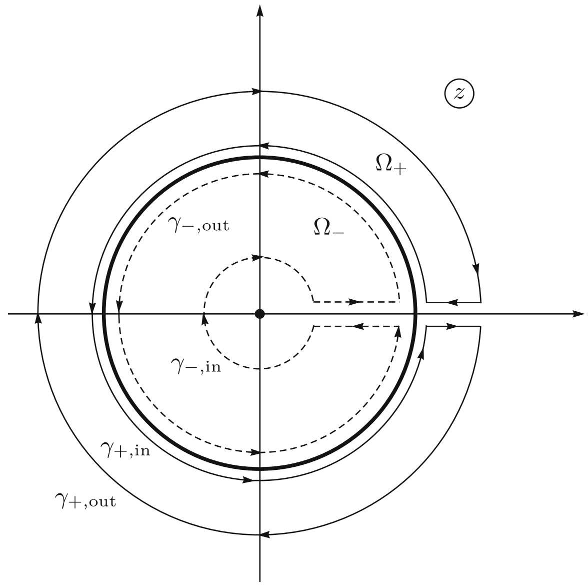

Fig. 1. The continuous spectrum Ω of $L_n(z)$ (bold circle) and the integration contours.

If $a^{\pm}(z)$ have respective zeros at $z_j^{\pm}$, then we can write

$$\varphi_n^{\pm} = \pm(z_j^{\pm})^{\mp 2n} b_j^{\pm} \xi_n^{\pm}, \tag{40}$$

and this leads to the integral representation for $\xi_n^{+}(z)$

$$\xi_n^{+}(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \oint_{\Omega} \frac{d\omega}{\omega - z} \omega^{2n} \rho^{-}(\omega) \xi_n^{-}(\omega) - \\ - \sum_{j=1}^{S} C_j^{-}(z_j^{-})^{2n} \left[\frac{\xi_n^{-}(z_j^{-})}{z - z_j^{-}} + \frac{\xi_n^{-}(-z_j^{-})}{z + z_j^{-}} \right]. \tag{41a}$$

Similarly, we can find an integral representation for $\xi_n^{-}(z)$ with $z \in \Omega_-$ by evaluating the integral $\mathcal{J}_{1,n}$,

$$\xi_n^{-}(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2\pi i} \oint_{\Omega} \frac{d\omega}{\omega - z} \omega^{-2n} \rho^{+}(\omega) \xi_n^{+}(\omega) + \\ + \sum_{j=1}^{S} C_j^{+}(z_j^{+})^{-2n} \left[\frac{\xi_n^{+}(z_j^{+})}{z - z_j^{+}} + \frac{\xi_n^{+}(-z_j^{+})}{z + z_j^{+}} \right]. \tag{41b}$$

The integrands in integral representations (41a) and (41b) are expressed in terms of $\rho^{\pm}(z)$ on the continuous spectrum only, while the sums that give the contributions from the discrete spectrum are expressed in terms of the normalization constants $C_j^{\pm}(z_j^{\pm})$. Hence, the system of singular integral equations (40) and (41) admits a unique solution, and the minimum set of scattering data $\mathcal{F}_1 = \{\rho^{+}(z), \rho^{-}(z), z \in |z| = 1,\ z_j^{\pm},\ j = 1, \dots, S\}$ contains all information needed to uniquely recover the Jost solutions $\xi_n^{\pm}(z)$.

4.1. One-soliton solution. In the case where the Lax operator $L_n(z)$ comprises proper eigenvalues but $\rho^{+}(z) = \rho^{-}(z) = 0$ on the continuous spectrum of $L_n(z)$, system (40), (41) reduces to a linear system of algebraic equations:

$$\xi_n^+(z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{j=1}^{S} C_j^-(z_j^-)^{2n}\left[\frac{\xi_n^-(z_j^-)}{z - z_j^-} + \frac{\xi_n^-(-z_j^-)}{z + z_j^-}\right], \tag{42a}$$

$$\xi_n^-(z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{S} C_j^+(z_j^+)^{-2n}\left[\frac{\xi_n^+(z_j^+)}{z - z_j^+} + \frac{\xi_n^+(-z_j^+)}{z + z_j^+}\right]. \tag{42b}$$

Respectively setting $z = \pm z_j^{\pm}$ in (42a) and (42b) gives

$$\begin{aligned}
\xi_n^+(\pm z_j^+) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mp \sum_{k=1}^{S} C_k^-(z_k^-)^{2n}\left[\frac{\xi_n^-(z_k^-)}{z_j^+ \mp z_k^-} + \frac{\xi_n^-(-z_k^-)}{z_j^+ \pm z_k^-}\right], \\
\xi_n^-(\pm z_j^-) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \sum_{k=1}^{S} C_k^+(z_k^+)^{-2n}\left[\frac{\xi_n^+(z_k^+)}{z_j^- \mp z_k^+} + \frac{\xi_n^+(-z_k^+)}{z_j^- \pm z_k^+}\right].
\end{aligned} \tag{43}$$

These relations show that

$$\begin{aligned}
\xi_n^{+,1}(-z_j^+) = -\xi_n^{+,1}(z_j^+) \quad \text{iff } \xi_n^{-,1}(-z_j^-) = \xi_n^{-,1}(z_j^-), \\
\xi_n^{+,2}(-z_j^+) = \xi_n^{+,2}(z_j^+) \quad \text{iff } \xi_n^{-,2}(-z_j^-) = -\xi_n^{-,2}(z_j^-).
\end{aligned} \tag{44}$$

It then follows that the solution of (42a) and (42b) is

$$\xi_n^{-,1}(z_1^-) = \left[1 - 4C_1^+C_1^- \frac{(z_1^+)^{-2(n-1)}(z_1^-)^{2n}}{((z_1^+)^2 - (z_1^-)^2)^2}\right]^{-1}, \tag{45}$$

$$\xi_n^{+,2}(z_1^+) = \left[1 + 4C_1^+C_1^- \frac{(z_1^-)^{2(n-1)}(z_1^+)^{-2n}}{((z_1^+)^2 - (z_1^-)^2)^2}\right]^{-1}. \tag{46}$$

Moreover, the potentials are given by

$$Q_{n-1}^+ = -2C_1^+(z_1^+)^{-2n-2}\xi_n^{+,(2)}(z_1^+), \tag{47}$$

$$Q_n^- = 2C_1^-(z_1^-)^{2n-2}\xi_n^{-,(1)}(z_1^-). \tag{48}$$

Substituting Eqs. (45) and (46) in (47) and (48) then gives the general form of a one-soliton solution

$$Q_{1n}^+ = -\frac{2C_1^-(z_1^-)^{2n}}{1 + 4C_1^+C_1^-((z_1^+)^2 - (z_1^-)^2)^{(-2)}(z_1^+)^{-2n}(z_1^-)^{2(n+1)}}, \tag{49}$$

$$Q_{1n}^- = \frac{2C_1^+(z_1^+)^{-2(n+1)}}{1 + 4C_1^+C_1^-((z_1^+)^2 - (z_1^-)^2)^{-2}(z_1^+)^{-2n}(z_1^-)^{2(n+1)}}. \tag{50}$$

Taking the time evolution of the normalization constants

$$C_1^+(z,\tau) = C_1^+(0)e^{2i\omega_1^+\tau}, \qquad C_1^-(z,\tau) = C_1^-(0)e^{-2i\omega_1^-\tau},$$

where $\omega^{\pm} = (i/2)(z_1^{\pm} - (z_1^{\pm})^{-1})^2$, and the canonical symmetry condition $Q_n^- = -(Q_{-n}^+)^*$ into account, we can reduce (49) and (50) to the standard form of the one-soliton solution of the nonlocal AL equation:

$$Q^+_{1n} = \frac{(z^-_1 z^+_1)^{-1}((z^+_1)^2 - (z^-_1)^2)e^{i\alpha^-_1} e^{-2i\omega^-_1\tau}(z^-_1)^{2n-1}}{1 + e^{i(\alpha^+_1+\alpha^-_1)}e^{2i(\omega^+_1-\omega^-_1)\tau}(z^+_1)^{-2n}(z^-_1)^{2n}}. \tag{51}$$

This reproduces the result obtained by Ablowitz and Musslimani.

4.2. Two-soliton solution. Similarly (as for one pole), we obtain the two-soliton solutions from a singular RHP with a quartet of discrete eigenvalues (singularities) $z^+_{\{1,2\}}$ and $z^-_{\{1,2\}}$. The starting point here are the linear integral equations for $\xi^\pm_n(z)$:

$$\begin{aligned}\xi^+_n(z) = {}&\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2\pi i}\oint_\Omega\left(\frac{d\omega}{\omega - z}(\omega)^{2n}\rho^-(\omega)\xi^-_n(\omega)\right) - \\ &- \left[C^-_1(z^-_1)^{2n}\left(\frac{\xi^-_n(z^-_1)}{z - z^-_1} + \frac{\xi^-_n(-z^-_1)}{z + z^-_1}\right)\right] + \left[C^-_2(z^-_2)^{2n}\left(\frac{\xi^-_n(z^-_2)}{z - z^-_2} + \frac{\xi^-_n(-z^-_2)}{z + z^-_2}\right)\right],\\ \xi^-_n(z) = {}&\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2\pi i}\oint_\Omega\left(\frac{d\omega}{\omega - z}(\omega)^{-2n}\rho^+(\omega)\xi^+_n(\omega)\right) + \\ &+ \left[C^+_1(z^+_1)^{-2n}\left(\frac{\xi^+_n(z^+_1)}{z - z^+_1} + \frac{\xi^+_n(-z^+_1)}{z + z^+_1}\right)\right] + \left[C^+_2(z^+_2)^{-2n}\left(\frac{\xi^+_n(z^+_2)}{z - z^+_2} + \frac{\xi^+_n(-z^+_2)}{z + z^+_2}\right)\right].\end{aligned} \tag{52}$$

The two-soliton solution again corresponds to zero reflection coefficients (i.e., $\rho^+(z) = \rho^-(z) = 0$ on $|z| = 1$). In this case, system (52) reduces to a linear system of algebraic equations for $\xi^+_n(z)$,

$$\begin{aligned}\xi^+_n(z) = {}&\begin{pmatrix}0\\1\end{pmatrix} - \left[C^-_1(z^-_1)^{2n}\left(\frac{\xi^-_n(z^-_1)}{z - z^-_1} + \frac{\xi^-_n(-z^-_1)}{z + z^-_1}\right)\right] - \\ &- \left[C^-_2(z^-_2)^{2n}\left(\frac{\xi^-_n(z^-_2)}{z - z^-_2} + \frac{\xi^-_n(-z^-_2)}{z + z^-_2}\right)\right],\end{aligned} \tag{53a}$$

and a similar system for $\xi^-_n(z)$,

$$\begin{aligned}\xi^-_n(z) = {}&\begin{pmatrix}1\\0\end{pmatrix} + \left[C^+_1(z^+_1)^{-2n}\left(\frac{\xi^+_n(z^+_1)}{z - z^+_1} + \frac{\xi^+_n(-z^+_1)}{z + z^+_1}\right)\right] + \\ &+ \left[C^+_2(z^+_2)^{-2n}\left(\frac{\xi^+_n(z^+_2)}{z - z^+_2} + \frac{\xi^+_n(-z^+_2)}{z + z^+_2}\right)\right].\end{aligned} \tag{53b}$$

Here, $\xi^+_n(\pm z^+_j)$ are the FAS $\xi^+_n(z)$ evaluated at the eigenvalue $\pm z^+_j$ (similarly, $\xi^-_n(\pm z^-_j)$ are the FAS $\xi^-_n(z)$ evaluated at the eigenvalue $\pm z^-_j$). We can find the expressions for these vectors by evaluating (53a) at the points $\pm z^+_{\{1,2\}}$ and (53b) at the points $\pm z^-_{\{1,2\}}$. This results in a linear algebraic system composed of (53a) and (53b):

$$\begin{aligned}\xi^+_n(\pm z^+_1) = {}&\begin{pmatrix}0\\1\end{pmatrix} - \left[C^-_1(z^-_1)^{2n}\left(\frac{\xi^-_n(z^-_1)}{\pm z^+_1 - z^-_1} + \frac{\xi^-_n(-z^-_1)}{\pm z^+_1 + z^-_1}\right)\right] - \\ &- \left[C^-_2(z^-_2)^{2n}\left(\frac{\xi^-_n(z^-_2)}{\pm z^+_1 - z^-_2} + \frac{\xi^-_n(-z^-_2)}{\pm z^+_1 + z^-_2}\right)\right],\end{aligned}$$

$$
\begin{aligned}
\xi_n^+(\pm z_2^+) = & \begin{pmatrix}0\\1\end{pmatrix} - \left[C_1^-(z_1^-)^{2n}\left(\frac{\xi_n^-(z_1^-)}{\pm z_2^+ - z_1^-} + \frac{\xi_n^-(-z_1^-)}{\pm z_2^+ + z_1^-}\right)\right] - \\
& - \left[C_2^-(z_2^-)^{2n}\left(\frac{\xi_n^-(z_2^-)}{\pm z_2^+ - z_2^-} + \frac{\xi_n^-(-z_2^-)}{\pm z_2^+ + z_2^-}\right)\right], \\
\xi_n^-(\pm z_1^-) = & \begin{pmatrix}1\\0\end{pmatrix} + \left[C_1^+(z_1^+)^{-2n}\left(\frac{\xi_n^+(z_1^+)}{\pm z_1^- - z_1^+} + \frac{\xi_n^+(-z_1^+)}{\pm z_1^- + z_1^+}\right)\right] + \\
& + \left[C_2^+(z_2^+)^{-2n}\left(\frac{\xi_n^+(z_2^+)}{\pm z_1^- - z_2^+} + \frac{\xi_n^+(-z_2^+)}{\pm z_1^- + z_2^+}\right)\right], \\
\xi_n^-(\pm z_2^-) = & \begin{pmatrix}1\\0\end{pmatrix} + \left[C_1^+(z_1^+)^{-2n}\left(\frac{\xi_n^+(z_1^+)}{\pm z_2^- - z_1^+} + \frac{\xi_n^+(-z_1^+)}{\pm z_2^- + z_1^+}\right)\right] + \\
& + \left[C_2^+(z_2^+)^{-2n}\left(\frac{\xi_n^+(z_2^+)}{\pm z_2^- - z_2^+} + \frac{\xi_n^+(-z_2^+)}{\pm z_2^- + z_2^+}\right)\right].
\end{aligned} \tag{54}
$$

It follows from (54) that

$$
\begin{aligned}
\xi_n^{+,1}(-z_j^+) = -\xi_n^{+,1}(z_j^+) \quad \text{iff } \xi_n^{-,1}(-z_j^-) = \xi_n^{-,1}(z_j^-), \\
\xi_n^{+,2}(-z_j^+) = \xi_n^{+,2}(z_j^+) \quad \text{iff } \xi_n^{-,2}(-z_j^-) = -\xi_n^{-,2}(z_j^-).
\end{aligned} \tag{55}
$$

We can recover Q_n^- from the power series expansion of the right-hand side of $\xi_n^{-,2}(z)$ in (53b) and take the residue at $z \to z_1^+$ or at $z \to z_2^+$, whence we can then obtain

$$
Q_n^- = -\frac{1}{2}C_1^+(z_1^+)^{-2n-2}\xi_n^{+,2}(z_1^+) + \frac{2C_2^+(z_2^+)^{-2n}}{(z_1^+)^2 - (z_2^+)^2}\xi_n^{+,2}(z_2^+) \tag{56}
$$

or

$$
Q_n^- = \frac{2C_1^+(z_1^+)^{-2n}}{(z_2^+)^2 - (z_1^+)^2}\xi_n^{+,2}(z_1^+) - \frac{1}{2}C_2^+(z_2^+)^{-2n-2}\xi_n^{+,2}(z_2^+). \tag{57}
$$

But it is difficult to find the potential from $\varphi_n^-(z)$. To fix the problem, we multiply the Laurent expansion function $\chi_n^-(z)$ by $\left(\begin{smallmatrix}1 & 0\\ 0 & c_n\end{smallmatrix}\right)$ and compare it with the right-hand side of (53a):

$$
(\tilde{\xi}_n^-, \widetilde{\varphi}_n^-) \simeq \widetilde{\chi}_n^-(z) = \begin{pmatrix}1 & 0\\ 0 & c_n\end{pmatrix}\begin{pmatrix}c_n^{-1} & zQ_{n-1}^+\\ c_n^{-1}zQ_n^- & 1\end{pmatrix} = \begin{pmatrix}c_n^{-1} & zQ_{n-1}^+\\ zQ_n^- & c_n\end{pmatrix}, \tag{58}
$$

where $\widetilde{\chi}_n^-(z)$ has the same power series expansions as $\chi_n^-(z)$. We can then find the potential Q_{n-1}^+ from $\widetilde{\varphi}_n^{-,1}$ in (58) because from (36) (for $b^-(z) = 0$),

$$
\begin{aligned}
\widetilde{\varphi}_n^-(z) = & \begin{pmatrix}0\\1\end{pmatrix} - \left[C_1^-(z_1^-)^{2n}\left(\frac{\xi_n^-(z_1^-)}{z - z_1^-} + \frac{\xi_n^-(-z_1^-)}{z + z_1^-}\right)\right] - \\
& - \left[C_2^-(z_2^-)^{2n}\left(\frac{\xi_n^-(z_2^-)}{z - z_2^-} + \frac{\xi_n^-(-z_2^-)}{z + z_2^-}\right)\right].
\end{aligned} \tag{59}
$$

We have a quartet of eigenvalues $\pm z_j^+$ and $\pm z_j^-$ respectively with $|z_j^+| > 1$ and $|z_j^-| < 1$ and can therefore solve linear algebra system (54) for $\xi_n^-(z_1^-)$, $\xi_n^-(z_2^-)$, $\xi_n^+(z_1^+)$, and $\xi_n^+(z_2^+)$. In particular, to find Q_n^+, we need $\xi_n^-(z_1^-)$ and $\xi_n^-(z_2^-)$:

$$\xi_n^{+,1}(z_1^+) = -2z_1^+\left[\frac{C_1^-(z_1^-)^{2n}}{(z_1^+)^2-(z_1^-)^2}\xi_n^{-,1}(z_1^-) + \frac{C_2^-(z_2^-)^{2n}}{(z_1^+)^2-(z_2^-)^2}\xi_n^{-,1}(z_2^-)\right], \tag{60a}$$

$$\xi_n^{+,1}(z_2^+) = -2z_2^+\left[\frac{C_1^-(z_1^-)^{2n}}{(z_2^+)^2-(z_1^-)^2}\xi_n^{-,1}(z_1^-) + \frac{C_2^-(z_2^-)^{2n}}{(z_2^+)^2-(z_2^-)^2}\xi_n^{-,1}(z_2^-)\right], \tag{60b}$$

$$\xi_n^{-,1}(z_1^-) = 1 + \frac{2C_1^+(z_1^+)^{-2(n-1)}}{(z_1^-)^2-(z_1^+)^2}\xi_n^{+,1}(z_1^+) + \frac{2C_2^+(z_2^+)^{-2(n-1)}}{(z_1^-)^2-(z_2^+)^2}\xi_n^{+,1}(z_2^+), \tag{60c}$$

$$\xi_n^{-,1}(z_2^-) = 1 + \frac{2C_1^+(z_1^+)^{-2(n-1)}}{(z_2^-)^2-(z_1^+)^2}\xi_n^{+,1}(z_1^+) + \frac{2C_2^+(z_2^+)^{-2(n-1)}}{(z_2^-)^2-(z_2^+)^2}\xi_n^{+,1}(z_2^+). \tag{60d}$$

Comparing the power series expansion of the right-hand side of (59) and expansion (58), we now obtain the potential for $z \to z_1^-$ or $z \to z_2^-$

$$Q_{n-1}^+ = \frac{1}{2}C_1^-(z_1^-)^{2n-2}\xi_n^{-,1}(z_1^-) - \frac{2C_2^-(z_2^-)^{2n}}{(z_1^-)^2-(z_2^-)^2}\xi_n^{-,1}(z_2^-) \tag{61}$$

or

$$Q_{n-1}^+ = \frac{-2C_1^-(z_1^-)^{2n}}{(z_2^-)^2-(z_1^-)^2}\xi_n^{-,1}(z_1^-) + \frac{1}{2}C_2^-(z_2^-)^{2n-2}\xi_n^{-,1}(z_2^-). \tag{62}$$

Substituting each (60c) and (60d) in (61) or (62) and using involution (4) with the minus sign, we then find that Eq. (61) solves the nonlocal discrete NLS equation.

5. Spectral properties of $L_n(z)$ and completeness of the Jost solutions

The crucial fact that determines the spectral properties of the operator $L_n(z)$ is the choice of the class of functions from which we choose the potential $Q_n(t)$. Here, we assume that $Q_n(t)$ is a differentiable function for all $t \in \mathbb{R}$ and exists for all $n \in \mathbb{Z}$. In addition, we assume that it tends to zero as $n \to \pm\infty$.

The FAS $\chi_n^\pm(z)$ of $L_n(z)$ allows constructing the resolvent $R_{n,m}(z)$ of the operator $L_n(z)$ and then investigating its spectral properties. From the general theory of linear operators, we know that the point in the complex-z plane is a regular point if $R_{n,m}(z)$ is a bounded integral operator. In each connected subset of regular points, $R(z)$ is analytic in z. The points that are not regular constitute the spectrum of $L_n(z)$: the continuous spectrum of $L_n(z)$ consists of all points z for which $R_{n,m}(z)$ is an unbounded integral operator, while the discrete spectrum of $L_n(z)$ consists of all points z for which $R_{n,m}(z)$ develops pole singularities.

The kernel of the resolvent $R_{n,m}(z)$ can be expressed in terms of the FAS of $\chi_n^\pm(z)$ as

$$R_{\{n,m\}}^+(z) = \chi_{n+1}^+(z)\begin{pmatrix}\theta(m-n) & 0\\ 0 & \theta(n-m)\end{pmatrix}\widehat{\chi}_m^+(z), \tag{63a}$$

$$R_{\{n,m\}}^-(z) = \chi_{n+1}^-(z)\begin{pmatrix}\theta(n-m) & 0\\ 0 & \theta(m-n)\end{pmatrix}\widehat{\chi}_m^-(z). \tag{63b}$$

We note that $R_{\{n,m\}}^\pm(z)$ are by construction respectively analytic in $\gamma^\pm$.

We derive the completeness relation for the Jost solutions of L_n by constructing the partition of unity of the group of fundamental solutions of $L_n(z)$. For this, we again use the contour integration method for suitably chosen contours that do not cross the continuous spectrum of $L_n(z)$. We must evaluate the integral

$$\mathcal{J}_{R,\{n,m\}}(z) = \frac{1}{2\pi i}\left(\oint_{\gamma^+} dz\, R_n^+(z) - \oint_{\gamma^-} dz\, R_n^-(z)\right) \tag{64}$$

along the contours $\gamma_\pm$, as shown in Fig. 1. According to Cauchy's residue theorem, we have

$$\mathcal{J}_{R,\{n,m\}}(z) = \sum_{j=1}^{S}\left(\operatorname*{Res}_{z=\pm z_j^+} R_n^+(z) + \operatorname*{Res}_{z=\pm z_j^-} R_n^-(z)\right). \tag{65}$$

Here, $z_j^\pm$ denotes the discrete eigenvalues of $L_n(z)$ that are respectively outside or inside the unit circle $|z| = 1$. We again assume that their numbers are equal and that all of them are isolated singularities for the resolvent.

Each of the integrals in (64) can be written as a sum of integrals over the continuous spectrum Ω and an integral involving the asymptotic forms as $|z| \to \infty$ and $|z| \to 0$:

$$\frac{1}{2\pi i}\oint_{\gamma^\pm} dz\, R_n^\pm(z) = \int_{|z|=1} dz\, R_n^\pm(z) - \oint_{\gamma^\pm_{\rm as}} dz\, R_n^\pm(z). \tag{66}$$

First, to find the residues in (65), we need the Laurent series expansions of the FAS and scattering data around the points of the discrete spectrum $z_j^\pm$. Using (20) and (21), we can write the expansions

$$\begin{aligned}
&a^\pm(z) = (z-(\pm z_j^\pm))\dot a_j^\pm + \frac{1}{2}(z-(\pm z_j^\pm))^2 \ddot a_j^\pm + \ldots,\\
&\alpha^\pm(z) = (z-(\pm z_j^\pm))\dot \alpha_j^\pm + \frac{1}{2}(z-(\pm z_j^\pm))^2 \ddot \alpha_j^\pm + \ldots,\\
&\chi_n^+(z_j^+) = \psi_{n,j}^+(z)(b_j^+,1) = \phi_{n,j}^+(z)(1,1/b_j^+),\\
&\widehat{\chi}_n^+(z_j^+) = \begin{pmatrix}1\\ -\beta_j^+\end{pmatrix}\frac{\widetilde{\Psi}_{n,j}^+(z)}{(z-(\pm z_j^+))\dot\alpha_j^+} = \begin{pmatrix}1/\beta_j^+\\ -1\end{pmatrix}\frac{\widetilde{\Phi}_{n,j}^+(z)}{(z-(\pm z_j^+))\dot\alpha_j^+},\\
&\chi_n^-(z_j^-) = \psi_{n,j}^-(z)(1,-b_j^-) = \phi_{n,j}^-(z)(-1/b_j^-,1),\\
&\widehat{\chi}_n^-(z_j^-) = -\begin{pmatrix}\beta_j^-\\ 1\end{pmatrix}\frac{\widetilde{\Psi}_{n,j}^-(z)}{(z-(\pm z_j^-))\dot\alpha_j^-} = \begin{pmatrix}1\\ 1/\beta_j^-\end{pmatrix}\frac{\widetilde{\Phi}_{n,j}^-(z)}{(z-(\pm z_j^-))\dot\alpha_j^-},
\end{aligned} \tag{67}$$

where $\widetilde{\Psi}_n(z)$ and $\widetilde{\Phi}_n(z)$ are respectively related to $\psi_n(z)$ and $\phi_n(z)$. As a result, we can find the residues of $R_n^\pm(z)$ at $z = \pm z_j^\pm$:

$$\operatorname*{Res}_{z=\pm z_j^\pm} R_{n,m}^\pm(z) = \mp\frac{\phi_{n+1,j}^\pm(z)\widetilde{\Psi}_{m,j}^\pm(z)}{\dot\alpha_j^\pm(z)}. \tag{68}$$

We can now calculate the jump of $R_n(z)$ on the unit circle $|z| = 1$. Using (63a), we can obtain the result

$$\int_{|z|=1} dz\,(R_{n,m}^+(z) - R_{n,m}^-(z)) = \int_{|z|=1} dz\left(\frac{\phi_{n+1}^+(z)\widetilde{\Psi}_m^+(z)}{\alpha^+(z)} + \frac{\phi_{n+1}^-(z)\widetilde{\Psi}_m^-(z)}{\alpha^-(z)}\right). \tag{69}$$

Finally, we must calculate the contribution of the integrals over the asymptotic circles (i.e., as $|z| \to \infty$ and $|z| \to 0$). For this, we need the asymptotic forms of the FAS $\chi_{n,as}^\pm(z)$ as $z \to \infty$ and $z \to 0$:

$$\chi_{\{\mathrm{as},n\}}^+(z) = \begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix} + \mathcal{O}(1/z), \quad z \to \infty,$$

$$\chi_{\{\mathrm{as},n\}}^-(z) = \begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix} + \mathcal{O}(z), \qquad z \to 0.$$

A direct contour integration shows (if we take a limit $|z| \to \infty$ in the integral over $\gamma_{\rm as}^-$ and a limit $|z| \to 0$ in the integral over $\gamma_{\rm as}^+$) that

$$\frac{1}{2\pi i}\left(\oint_{\gamma_{\rm as}^+} dz\, R^+_{\{{\rm as},n,m\}} + \oint_{\gamma_{\rm as}^-} dz\, R^-_{\{{\rm as},n,m\}}\right) = \delta(n-m)\mathbb{I}. \tag{70}$$

As a result, combining (64) and (65) and taking (68), (69), and (70) into account, we obtain the completeness relation

$$\begin{aligned}\delta(n-m)\mathbb{1} = \frac{1}{2\pi i}\int_{|z|=1} dz \left(\frac{\phi^+_{n+1}(z)\widetilde{\Psi}^+_m(z)}{\alpha^+(z)} + \frac{\phi^-_{n+1}(z)\widetilde{\Psi}^-_m(z)}{\alpha^-(z)}\right) + \\ + \sum_{j=1}^{S}\left(\frac{\phi^+_{\{n+1,j\}}(z)\widetilde{\Psi}^+_{m,j}(z)}{\dot{\alpha}^+_j(z)} - \frac{\phi^-_{\{n+1,j\}}(z)\widetilde{\Psi}^-_{m,j}(z)}{\dot{\alpha}^-_j(z)}\right).\end{aligned} \tag{71}$$

Therefore, the Jost solutions $\phi^\pm_n(z)$ form a complete set of functions over the space of fundamental solutions of $L_n(z)$.

Based on completeness relation (71), we can expand every function $Y(z)$ from the space of solutions of $L_n(z)$ over the complete set of Jost solutions by the expansion formulas

$$\begin{aligned}Y(z) = \frac{1}{2\pi i}\int_{|z|=1} dz \left(\phi^+_{n+1}(z)y^+_m(z) + \phi^-_{n+1}(z)y^-_m(z)\right) + \\ + \sum_{j=1}^{S}\left(\phi^+_{\{n+1,j\}}(z)y^+_{\{m,j\}}(z) - \phi^-_{\{n+1,j\}}(z)y^-_{\{m,j\}}(z)\right),\end{aligned} \tag{72}$$

where

$$\begin{aligned} y^\pm_m(z) &= \frac{1}{\alpha^\pm(z)}\int_{|z|=1} dz\, \widetilde{\Psi}^\pm_m(z)Y_m(z), \\ y^\pm_{\{m,j\}}(z) &= \frac{1}{\dot{\alpha}^\pm_j}\int_{|z|=1} dz\, \widetilde{\Psi}^\pm_{\{m,j\}}(z)Y_m(z).\end{aligned} \tag{73}$$

6. Conclusions

We have studied a nonlocal version [43] of the semidiscrete NLS equation in the AL form. This equation appears to be $\mathcal{PT}$ symmetric. We formulated the direct scattering problem for the nonlocal AL equation. This included constructing the Jost solutions, the minimum set of scattering data, and the FAS. Based on the formulation of the inverse scattering transform for (4) in the form of an additive Riemann–Hilbert boundary value problem, we then derived the one- and two-soliton solutions.

It was shown in [43] that the one-soliton solution develops a singularity in a finite time. This was due to the disbalance of the associated RHP: the numbers of zeros of the FAS inside the boundary contour is not equal to the number of zeros inside the contour. The nonlocal involution requires that if z_j is a discrete eigenvalue, then z_j^* must also be an eigenvalue, i.e., both z_j and z_j^* must be either inside or outside the unit circle. Depending on the positions of the discrete eigenvalues $z_j^\pm$ in the spectral plane, there are two regimes for the two-soliton solution: if one of the discrete eigenvalues is inside the unit circle and the other is outside, then the nonlocal involution preserves their number balanced inside and outside the contour, and the corresponding two-soliton solutions are consequently regular for all t. Otherwise, the two-soliton solution again develops a singularity in a finite time.

Finally, we briefly outlined the spectral properties of the Lax operator $L_n(z)$. We derived the completeness relations for the Jost solutions and obtained expansions over the complete set of Jost solutions for a generic function from the space of solutions of $L_n(z)$.

The results in this paper can be extended in several directions:

1. To construct a gauge-covariant formulation of the ISM for nonlocal AL equation (49) including the generating (recursion) operator [57] and its spectral decomposition [55], the description of the class of the differential–difference equations solvable by spectral problem (6) (i.e., the corresponding integrable hierarchy), and the description of the infinite set of integrals of motion and the hierarchy of Hamiltonian structures.

2. To study gauge-equivalent systems [11]–[13].

3. To study the ISM for the equivalent spectral eigenvalue problem (12). In this case, the corresponding RHP is with a canonical normalization.

4. To study the associated Darboux transformations and their generalizations for both local and nonlocal AL equations. This would provide an algebraic method for constructing and classifying possible soliton solutions, also including rational solutions [58].

5. To extend the results of this paper to the case of nonvanishing boundary conditions (a nontrivial background) [59]–[62]. In the local case, such solutions are of interest in nonlinear optics: they arise in the theory of ultrashort femtosecond nonlinear pulses in optical fibers. The nonlocal reduction of the AL equation can be of particular interest in the theory of electromagnetic waves in artificial heterogeneous media [48]. The considerations required in this case are more complicated and will be discussed elsewhere.

6. To study multicomponent generalizations [10], [14], [15], [56], [63], [64] for both the local and the nonlocal semidiscrete NLS equation. This includes the block AL system [26] and generalizations to homogeneous and symmetric spaces. Such multicomponent generalizations are much more complicated than in the continuous case and, to the best of our knowledge, they have not yet been studied.

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