Theoretical and Mathematical Physics, **197**(1): 1430–1450 (2018)

NONLOCAL REDUCTIONS OF THE MULTICOMPONENT NONLINEAR SCHRÖDINGER EQUATION ON SYMMETRIC SPACES

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Our aim is to develop the inverse scattering transform for multicomponent generalizations of nonlocal reductions of the nonlinear Schrödinger (NLS) equation with $\mathcal{PT}$ symmetry related to symmetric spaces. This includes the spectral properties of the associated Lax operator, the Jost function, the scattering matrix, the minimum set of scattering data, and the fundamental analytic solutions. As main examples, we use the Manakov vector Schrödinger equation (related to **A.III***-symmetric spaces) and the multicomponent NLS (MNLS) equations of Kulish–Sklyanin type (related to* **BD.I***-symmetric spaces). Furthermore, we obtain one- and two-soliton solutions using an appropriate modification of the Zakharov–Shabat dressing method. We show that the MNLS equations of these types admit both regular and singular soliton configurations. Finally, we present different examples of one- and two-soliton solutions for both types of models, subject to different reductions.*

Keywords: integrable system, multicomponent nonlinear Schrödinger equation, Lax representation, Zakharov–Shabat system, spectral decompositions, $\mathcal{PT}$ symmetry, inverse scattering transform, Riemann–Hilbert problem, dressing method

DOI: 10.1134/S0040577918100033

1. Introduction

One of the most important and popular completely integrable nonlinear partial differential equations (PDEs) is the nonlinear Schrödinger (NLS) equation [1]–[4]:

$$iq_t + q_{xx} + 2|q|^2 q(x,t) = 0. \tag{1}$$

Here, $q(x,t)$ is a complex-valued function decaying sufficiently fast as $|x| \to \infty$ [4], [5]. It has been derived as a governing equation describing processes and phenomena in such diverse fields as deep water waves, plasma physics and nonlinear fibre optics. For instance, in optics, the NLS equation models wave propagation in Kerr media, where the nonlinearity is proportional to the field intensity.

Equation (1) appeared in the early stage of the development of the inverse scattering method (ISM) and the theory of solitons [2]–[7] and exhibits all remarkable properties of PDEs and systems of PDEs integrable by the ISM: it admits soliton solutions, has an infinite set of integrals of motion and multi-Hamiltonian formulation, and so on [2], [4]. The key tool in studying integrability by the ISM is the existence of a Lax representation of the nonlinear evolution equation (NLEE), i.e., the NLEE can be represented as a

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Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 197, No. 1, pp. 45–67, October, 2018.

compatibility condition of two linear operators [1]–[4], [8]. The scattering problem for (1) is given by the Zakharov–Shabat (ZS) system (related to the $sl(2,\mathbb{C})$ algebra):

$$\begin{gathered} L\chi \equiv \left(i\,\frac{d}{dx} + q(x,t) - \lambda\sigma_3\right)\chi(x,t,\lambda) = 0, \\ q(x,t) = \begin{pmatrix} 0 & q^+ \\ q^- & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{gathered} \tag{2}$$

The first integrable multicomponent generalization of scalar NLS equation (1) is the Manakov vector NLS (VNLS) equation:

$$\begin{gathered} iq_{1,t} + q_{1,xx} + 2(|q_1|^2 + |q_2|^2)q_1(x,t) = 0, \\ iq_{2,t} + q_{2,xx} + 2(|q_1|^2 + |q_2|^2)q_2(x,t) = 0, \end{gathered} \tag{3}$$

where $q(x,t)$ is a two-component vector-valued function. It is associated with a scattering problem of the ZS type related to the algebra $sl(3,\mathbb{C})$. Manakov [9] proposed it as an asymptotic model for the propagation of an electric field in a waveguide. This system was subsequently derived as a key model for light propagation in optical fibers [10], [11].

Manakov model (3) can be generalized to n-component vectors (see [3]):

$$i\mathbf{v}_t + \mathbf{v}_{xx} + 2(\mathbf{v}^\dagger, \mathbf{v})\mathbf{v} = 0, \qquad \mathbf{v} = \mathbf{v}(x,t), \tag{4}$$

where $\mathbf{v}$ is an n-component complex-valued vector and $(\,\cdot\,,\,\cdot\,)$ is the standard scalar product. It is again integrable by the ISM [2]–[5].

The classical ZS system can be generalized to a matrix (multicomponent) form in several ways. One standard way to do this is to consider Lax operators taking values in a simple Lie algebra $\mathfrak{g}$:

$$L = i\,\partial_x + Q - \lambda \mathrm{J}. \tag{5}$$

Here, J is a constant element of the simple Lie algebra $\mathfrak{g}$ [12], [13]. Generalizations of the NLS equation to symmetric spaces related to a Lie algebra $\mathfrak{g}$ were proposed in [14]–[16]. This includes coupled NLS systems with Lax pairs related the symmetric spaces **A.III**, **C.I**, **D.III**, and **BD.I** types in the Cartan classification [12], [17]. As a special case in this class, we can obtain the Manakov VNLS equation and the one studied by Kulish and Sklyanin [18]. All these generalizations are solvable by the ISM [4], [14], [18].

As the rank r of the underlying simple Lie algebra $\mathfrak{g}$ increases, the corresponding generic NLEE (or systems) contain as many independent complex-valued functions as the number of all roots of $\mathfrak{g}$ [19]–[21]. They are solvable for any r, but their possible physical applications for large r seem unrealistic. But new integrable and physically useful NLEEs might still be extracted by imposing reductions on $L(\lambda)$, i.e., algebraic restrictions on $Q(x,t)$ diminishing the number of independent functions in them and the number of equations [22]. Of course, such restrictions must be compatible with the dynamics of the NLEEs [20], [21].

A nonlocal integrable equation of NLS type was recently proposed in [23],

$$iq_t + q_{xx} + V(x,t)q(x,t) = 0, \qquad V(x,t) = 2q(x,t)q^*(-x,t), \tag{6}$$

with $\mathcal{PT}$ symmetry due to the invariance of the so-called self-induced potential $V(x,t)$ under the combined action of the parity and time-reversal symmetries. In the same paper, the one-soliton solution for this model was derived, and it was shown that it develops singularities in a finite time. Soon after, nonlocal $\mathcal{PT}$-symmetric generalizations were found for the Ablowitz–Ladik model in [24]. All these models are integrable by the ISM [1], [25]–[27]. Such nonlocal reductions of the NLS equation and its multicomponent generalizations are particularly interesting with regard to applications in $\mathcal{PT}$-symmetric optics, especially in developing of the theory of electromagnetic waves in artificial heterogeneous media [28]–[31] (see, e.g., [32] for an up-to-date review).

The first pseudo-Hermitian Hamiltonian with a real spectrum historically is the $\mathcal{PT}$-symmetric Hamiltonian in [33]. Pseudo-Hermiticity here means that the Hamiltonian $\mathcal{H}$ commutes with the operators of spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$: $\mathcal{PTH} = \mathcal{HPT}$ [34], [35]. The action of these operators is defined as $\mathcal{P}\colon x \to -x$ and $\mathcal{T}\colon t \to -t$ [36], [37]. If we assume that the wave function is a scalar, then this leads to the action of the operator of spatial reflection on the space of states $\mathcal{P}\psi(x,t) = \psi(-x,t)$ and $\mathcal{T}\psi(x,t) = \psi^*(x,-t)$. As a result, the Hamiltonian and the wave function are $\mathcal{PT}$ symmetric if $\mathcal{H}(x,t) = \mathcal{H}^*(-x,-t)$ and $\psi(x,t) = \psi^*(-x,-t)$ [38]. Here we also use the fact that the parity operator $\mathcal{P}$ is linear and unitary while the time-reversal operator $\mathcal{T}$ is antilinear and antiunitary.

Our aim here is study nonlocal reductions and to derive the corresponding soliton solutions for multicomponent NLS (MNLS) models related to symmetric spaces of the **A**.**III** and **BD**.**I** types. We do this based on examples of the vector NLS equations related to symmetric spaces of the **A**.**III** $\simeq SU(s+p)/S(U(s)\otimes U(p))$ type and the Kulish–Sklyanin (KS) model related to symmetric spaces of the **BD**.**I** $\simeq SO(2r+1)/SO(2)\otimes SO(2r-1)$ type.

This paper is as structured follows. In Sec. 2, we outline the form of the Lax operators and the general form of the NLEEs and also the nonlocal symmetries (involutions) of interest. In Sec. 3, we present the direct scattering problem: the Jost solutions, the scattering matrix, and the minimum set of scattering data and the fundamental analytic solutions (FAS). Furthermore, in Sec. 4, based on an appropriate modification of the ZS dressing method, we derive one- and two-soliton solutions of the corresponding NLEE equation related to **A**.**III** and **BD**.**I** symmetric spaces and study the effect of nonlocal reductions on them.

2. Preliminaries

2.1. Lax pairs and the general form of the equations. We start with the generic Lax pair for the MNLS equations on symmetric spaces [39], which can be represented in the form

$$L\chi(x,t,\lambda) \equiv i\,\partial_x\chi + (Q(x,t) - \lambda \mathrm{J})\chi(x,t,\lambda) = 0, \qquad U(x,t,\lambda) = Q(x,t) - \lambda\mathrm{J}, \tag{7a}$$

$$M\chi(x,t,\lambda) \equiv i\,\partial_t\chi + (V_0(x,t) + \lambda V_1(x,t) - \lambda^2\mathrm{J})\chi(x,t,\lambda) = 0, \tag{7b}$$

$$V_1(x,t) = Q(x,t), \qquad V_0(x,t) = i\,\mathrm{ad}_{\mathrm{J}}^{-1}\,\frac{dQ}{dx} + \frac{1}{2}[\mathrm{ad}_{\mathrm{J}}^{-1}\,Q, Q(x,t)]. \tag{7c}$$

Here, $\chi(x,t,\lambda)$ are the eigenfunctions of the Lax operators, $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are chosen to take values in a simple Lie algebra $\mathfrak{g}$ of rank r, and $\chi(x,t,\lambda)$ belong to the corresponding Lie group $\mathfrak{G}$. Also, J is a constant element of the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ (which can be always chosen to be represented by a diagonal matrix), and $\lambda\in\mathbb{C}$ is a spectral parameter. On the **A**.**III** $\simeq SU(p+s)/S(U(p)\times U(s))$ symmetric spaces, the potential of $L(\lambda)$ given by (7a) is

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^+ \\ \mathbf{q}^- & 0 \end{pmatrix}, \qquad \mathrm{J} = \frac{2}{s+p}\begin{pmatrix} p\mathbb{1}_s & 0 \\ 0 & -s\mathbb{1}_p \end{pmatrix}, \tag{8}$$

where $\mathbf{q}^+(x,t)$ and $(\mathbf{q}^-)^{\mathrm{T}}(x,t)$ are $s\times p$ matrix-valued functions belonging to the simple Lie algebra $\mathfrak{g}$ and $\mathbb{1}_s$ and $\mathbb{1}_p$ are the $s\times s$ and $p\times p$ identity matrices, $s+p=n$.

The NLEE can be written as a compatibility condition

$$[L(\lambda), M(\lambda)] = 0 \tag{9}$$

of the two Lax operators (7a) and (7b). In particular, if $L(\lambda)$ and $M(\lambda)$ are related to $\mathbf{A.III} \simeq SU(p+1)/S(U(1)\otimes U(p))$-symmetric spaces [14], then explicit parameterization (8) of the **A.III**-symmetric spaces gives the system

$$i\mathbf{q}_t^+ + \mathbf{q}_{xx}^+ + 2\mathbf{q}^+\mathbf{q}^-\mathbf{q}^+(x,t) = 0, \tag{10}$$

$$-i\mathbf{q}_t^- + \mathbf{q}_{xx}^- + 2\mathbf{q}^-\mathbf{q}^+\mathbf{q}^-(x,t) = 0. \tag{11}$$

The particular choice $s=1$ and $p=2$ (also assuming the standard involution $\mathbf{q}^- = (\mathbf{q}^+)^*$ for the NLS-type models) corresponds to the well-known Manakov system [9]. Its generalizations to n-dimensional vectors $\mathbf{q}^\pm$ is called the vector NLS equation [4].

Another class of MNLS equations comprises the KS models, which are related to the symmetric spaces of the $\mathbf{BD.I} \simeq SO(2r+1)/SO(2)\otimes SO(2r-1)$ type [40]–[44]. The generic NLEEs of this class can be written as

$$\begin{aligned} &i\mathbf{q}_t^+ + \mathbf{q}_{xx}^+ + 2(\mathbf{q}^+,\mathbf{q}^-)\mathbf{q}^+ - (\mathbf{q}^+, s_0\mathbf{q}^+)s_0\mathbf{q}^- = 0,\\ &i\mathbf{q}_t^- - \mathbf{q}_{xx}^- - 2(\mathbf{q}^+,\mathbf{q}^-)\mathbf{q}^- + (\mathbf{q}^-, s_0\mathbf{q}^-)s_0\mathbf{q}^+ = 0, \end{aligned} \tag{12}$$

which is associated with $\mathfrak{g} \simeq so(2r+1,\mathbb{C})$ linear system (7a) where

$$Q(x,t) = \begin{pmatrix} 0 & (\mathbf{q}^+)^{\mathrm{T}} & 0 \\ \mathbf{q}^- & \mathbf{0} & s_0\mathbf{q}^+ \\ 0 & (\mathbf{q}^-)^{\mathrm{T}}s_0 & 0 \end{pmatrix}, \qquad \mathrm{J} = \mathrm{diag}(1,\mathbf{0},-1). \tag{13}$$

Here, $\mathbf{q}^\pm$ are $(2r-1)$-component vectors, and s_0 is the matrix that defines the orthogonal algebra $\mathbf{B}_r \simeq so(2r+1)$,

$$X \in so(2r+1) \quad \text{iff} \quad X + S_0X^{\mathrm{T}}S_0^{-1} = 0,$$

$$S_0 = \sum_{s=1}^{2r+1}(-1)^{s+1}E_{s,2r+2-s}^{(2r+1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{14}$$

where $E_{s,p}^{(2r+1)}$ is a $(2r+1)\times(2r+1)$ matrix with the elements $(E_{s,p}^{(2r+1)})_{ij} = \delta_{si}\delta_{pj}$.

2.2. Symmetries of the Lax operator and reductions. Mikhailov [22] introduced a systematic way to describe and classify the class of admissible reductions for a given Lax pair using the so-called reduction group. The reduction group is a finite group preserving the Lax representation, i.e., it ensures that the reduction constraints are automatically compatible with the evolution [20], [21], [45], [46]. The reduction constraints diminish the number of independent functions and the number of equations. Here, we restrict ourself to $\mathbb{Z}_2$ reductions of the two types

$$(\mathrm{A}): \quad U(x,t,\lambda) = B(U(x,t,\lambda^*))^\dagger B^{-1}, \tag{15}$$

$$(\mathrm{B}): \quad -U(x,t,\lambda) = B(U(-x,t,-\lambda^*))^\dagger B^{-1}, \tag{16}$$

where B is a constant block-diagonal matrix of the form $B = \left(\begin{smallmatrix} \mathbf{B}_+ & 0 \\ 0 & \mathbf{B}_- \end{smallmatrix}\right)$ for **A.III**-symmetric spaces. For symmetric spaces of type **BD.I**, it can be written in the form

$$B = \begin{pmatrix} \mathbf{B}_+ & 0 & 0 \\ 0 & \mathbf{B}_\pm & 0 \\ 0 & 0 & \mathbf{B}_- \end{pmatrix}. \tag{17}$$

Here, we also assume that the blocks $\mathbf{B}_+$ and $\mathbf{B}_-$ are nonsingular matrices. We also note that the involution in (16) results in nonlocal reduction conditions.

Recalling the standard block-structure on **A.III**-symmetric spaces in (8), we can write the reduction conditions for the matrix blocks explicitly:

$$\text{(A)}: \quad \mathbf{q}^-(x,t) = \mathbf{B}_-(\mathbf{q}^+(x,t))^\dagger(\mathbf{B}_+)^{-1}, \qquad \mathbf{q}^+(x,t) = \mathbf{B}_+(\mathbf{q}^-(x,t))^\dagger(\mathbf{B}_-)^{-1}, \tag{18}$$

$$\text{(B)}: \quad \mathbf{q}^-(x,t) = -\mathbf{B}_-(\mathbf{q}^+(-x,t))^\dagger(\mathbf{B}_+)^{-1}, \qquad \mathbf{q}^+(x,t) = -\mathbf{B}_+(\mathbf{q}^-(-x,t))^\dagger(\mathbf{B}_-)^{-1}. \tag{19}$$

Using the respective local and nonlocal involutions (18) and (19), we can write the resulting KS models. If we take $\mathbf{B}$ to be the identity matrix, then local involution (18) gives the equations

$$i\mathbf{q}_t^+ + \mathbf{q}_{xx}^+ + 2(\mathbf{q}^+(x,t),(\mathbf{q}^+(x,t))^*)\mathbf{q}^+ - (\mathbf{q}^+, s_0\mathbf{q}^+)s_0(\mathbf{q}^+(x,t))^* = 0, \tag{20}$$

and nonlocal involution (19) gives

$$i\mathbf{q}_t^+ + \mathbf{q}_{xx}^+ + 2(\mathbf{q}^+(x,t),(\mathbf{q}^+(-x,t))^*)\mathbf{q}^+ - (\mathbf{q}^+, s_0\mathbf{q}^+)s_0(\mathbf{q}^+(-x,t))^* = 0. \tag{21}$$

Example 1 (Manakov model)**.** If $\mathfrak{g} \simeq sl(3,\mathbb{C})$, $p = 1$, and $s = 2$, then $\mathbf{q}^+(x,t)$ and $\mathbf{q}^-(x,t)$ are two-component vector functions. In addition, if we apply a (local) reduction of type (A) with $B = \mathbb{I}$, then we reproduce the standard Manakov VNLS equation:

$$\begin{aligned} &-iq_{1,t} + q_{1,xx} + 2(|q_1(x,t)|^2 + |q_2(x,t)|^2)q_1(x,t) = 0, \\ &-iq_{2,t} + q_{2,xx} + 2(|q_1(x,t)|^2 + |q_2(x,t)|^2)q_2(x,t) = 0. \end{aligned} \tag{22}$$

Taking an involution of type (B) with $\mathbf{B}_+ = \mp 1$ and $\mathbf{B}_- = \operatorname{diag}(\pm 1, \pm 1)$, then we obtain the nonlocal reduction of the Manakov model

$$\begin{aligned} &-iq_{1,t} + q_{1,xx} + 2(q_1(x,t)q_1^*(-x,t) + q_2(x,t)q_2^*(-x,t))q_1(x,t) = 0, \\ &-iq_{2,t} + q_{2,xx} + 2(q_1(x,t)q_1^*(-x,t) + q_2(x,t)q_2^*(-x,t))q_2(x,t) = 0. \end{aligned} \tag{23}$$

Example 2 (KS model)**.** In the simplest case of Lax operators related to $SO(5)/SO(2) \otimes SO(3)$ ($\operatorname{rank}\mathfrak{g} = 2$), we can set $\mathbf{q}^+ = (q_{12}^+, q_{13}^+, q_{14}^+)$ and $\mathbf{q}^- = (q_{12}^-, q_{13}^-, q_{14}^-)$. After the standard involution of type (A) with $B = \mathbb{I}$ is also assumed, compatibility condition (9) leads to the three-component NLS system

$$\begin{aligned} &i(q_t^+)_{12} + (q_{xx}^+)_{12} + 2(|q_{12}|^2 + 2|q_{13}|^2)q_{12} + 2(q_{14}^+)^*(q_{13}^+)^2 = 0, \\ &i(q_t^+)_{13} + (q_{xx}^+)_{13} + 2(|q_{12}|^2 + |q_{13}|^2 + |q_{14}|^2)q_{13} + 2(q_{13}^+)^* q_{14}^+ q_{12}^+ = 0, \\ &i(q_t^+)_{14} + (q_{xx}^+)_{14} + 2(|q_{14}|^2 + 2|q_{13}|^2)q_{14} + 2(q_{12}^+)^*(q_{13}^+)^2 = 0. \end{aligned} \tag{24}$$

This appears to be a model describing $\mathcal{F}$=1 spinor Bose–Einstein condensates in a one-dimensional approximation [19], [44], [47], [48]. A similar system was presented in [49] using an equivalent reduction.

3. Direct scattering transform for $L(\lambda)$

3.1. Jost solutions and scattering matrix. The starting point here are the so-called Jost solutions, which are determined by their asymptotic forms as $|x| \to \infty$:

$$\lim_{x\to\infty} e^{i\lambda Jx}\psi(x,t,\lambda) = \mathbb{1}_{s+p}, \qquad \lim_{x\to-\infty} e^{i\lambda Jx}\phi(x,t,\lambda) = \mathbb{1}_{s+p}, \quad \lambda \in \mathbb{R}. \tag{25}$$

Along with these functions, we can also use Jost solutions "normalized to unity"

$$\xi(x,t,\lambda)=\psi(x,t,\lambda)e^{i\lambda \mathrm{J}x}, \qquad \varphi(x,t,\lambda)=\phi(x,t,\lambda)e^{i\lambda \mathrm{J}x}, \tag{26}$$

satisfying the equation

$$i\,\frac{d\xi}{dx}+Q(x,t)\xi(x,t,\lambda)-\lambda[\mathrm{J},\xi(x,t,\lambda)]=0 \tag{27}$$

if (25) satisfies (7a).

On the continuous spectrum of $L(\lambda)$, the two Jost solutions $\psi(x,t,\lambda)$ and $\phi(x,t,\lambda)$ are related via the scattering matrix $T(\lambda)$,

$$\phi(x,t,\lambda)=\psi(x,t,\lambda)T(\lambda), \quad \lambda\in\mathbb{R}. \tag{28}$$

The scattering matrix $T(\lambda)$ belongs to the Lie group $\mathfrak{G}$ corresponding to the Lie algebra/symmetric space of $L(\lambda)$. For symmetric spaces of the type **A.III** in the Cartan classification, it has the block structure

$$T(\lambda)=\begin{pmatrix}\mathbf{a}^+(\lambda) & -\mathbf{b}^-(\lambda)\\ \mathbf{b}^+(\lambda) & \mathbf{a}^-(\lambda)\end{pmatrix}, \tag{29}$$

where $\mathbf{a}^+(\lambda)$ and $\mathbf{a}^-(\lambda)$ are square matrices and $\mathbf{b}^+(\lambda)$ and $\mathbf{b}^-(\lambda)$ are rectangular matrices. The block structure of the inverse of $T(\lambda)$ can be written as

$$\widehat{T}(\lambda)=\begin{pmatrix}\mathbf{c}^-(\lambda) & \mathbf{d}^-(\lambda)\\ -\mathbf{d}^+(\lambda) & \mathbf{c}^+(\lambda)\end{pmatrix}, \qquad \widehat{T}(\lambda)=T^{-1}(\lambda), \tag{30}$$

where

$$\begin{aligned}\mathbf{c}^\pm(\lambda)&=\hat{\mathbf{a}}^\mp(\lambda)(\mathbb{1}+\rho^\pm\rho^\mp)^{-1}=(\mathbb{1}+\tau^\mp\tau^\pm)^{-1}\hat{\mathbf{a}}^\mp(\lambda),\\ \mathbf{d}^\pm(\lambda)&=\hat{\mathbf{a}}^\mp(\lambda)\rho^\pm(\lambda)(\mathbb{1}+\rho^\mp\rho^\pm)^{-1}=(\mathbb{1}+\tau^\mp\tau^\pm)^{-1}\tau^\mp(\lambda)\hat{\mathbf{a}}^\pm(\lambda).\end{aligned} \tag{31}$$

Here, $\rho^\pm(\lambda)$ and $\tau^\pm(\lambda)$ are the respective reflection and transmission coefficients:

$$\rho^\pm(\lambda)=\mathbf{b}^\pm\hat{\mathbf{a}}^\pm(\lambda)=\hat{\mathbf{c}}^\pm\mathbf{d}^\pm(\lambda), \qquad \tau^\pm(\lambda)=\hat{\mathbf{a}}^\pm\mathbf{b}^\mp(\lambda)=\mathbf{d}^\mp\hat{\mathbf{c}}^\pm(\lambda). \tag{32}$$

For symmetric spaces of **BD.I** type, the block structures of $T(t,\lambda)$ and its inverse become

$$T(t,\lambda)=\begin{pmatrix}m_1^+ & -\vec{\mathbf{B}}^{-T} & c_{(1)}^-\\ \vec{\mathbf{b}}^+ & \mathbf{T}_{22} & -s_0\vec{\mathbf{b}}^-\\ c_{(1)}^+ & \vec{\mathbf{B}}^{+T}s_0 & m_1^-\end{pmatrix}, \qquad \widehat{T}(t,\lambda)=\begin{pmatrix}m_1^- & \vec{\mathbf{b}}^{-T} & c_{(1)}^-\\ -\vec{\mathbf{B}}^+ & \widehat{\mathbf{T}}_{22} & s_0\vec{\mathbf{B}}^-\\ c_{(1)}^+ & -\vec{\mathbf{b}}^{+T}s_0 & m_1^+\end{pmatrix}, \tag{33}$$

where $\vec{\mathbf{B}}^\pm(t,\lambda)$ and $\vec{\mathbf{b}}^\pm(t,\lambda)$ are $(2r-1)$-component vectors and $c_{(1)}^\pm(\lambda)$ and $m_1^\pm(\lambda)$ are scalar functions. The reflection coefficients $\vec{\rho}^{\,\pm}(\lambda)$, the transmission coefficients $\vec{\tau}^{\,\pm}(\lambda)$, and the functions $c_{(1)}^\pm(\lambda)$ are determined by the generalized Gauss decomposition of the matrix

$$T(t,\lambda)=T_{\mathrm{J}}^-(t,\lambda)D_{\mathrm{J}}^+(t,\lambda)\widehat{S}_{\mathrm{J}}^+(t,\lambda)=T_{\mathrm{J}}^+(t,\lambda)D_{\mathrm{J}}^-(t,\lambda)\widehat{S}_{\mathrm{J}}^-(t,\lambda). \tag{34}$$

Here, $S_{\mathrm{J}}^\pm$ and $T_{\mathrm{J}}^\pm$ are upper- and lower-block-triangular matrices, which can be written in the forms

$$S_{\mathrm{J}}^\pm=e^{(\pm(\vec{\tau}^{\,\pm}(\lambda,t).\vec{E}_1^\pm))}, \qquad T_{\mathrm{J}}^\pm=e^{(\mp(\vec{\rho}^{\,\mp}(\lambda,t).\vec{E}_1^\pm))}, \tag{35}$$

where

$$\vec{\tau}^{\pm}(\lambda,t)=\frac{\vec{\mathbf{B}}^{\mp}}{m_1^{\pm}},\qquad \vec{\rho}^{\pm}(\lambda,t)=\frac{\vec{\mathbf{b}}^{\pm}}{m_1^{\pm}},\qquad c_{(1)}^{\pm}=\frac{m_1^{\pm}}{2}(\vec{\rho}^{\pm,\mathrm{T}},s_0\vec{\rho}^{\pm})=\frac{m_1^{\mp}}{2}(\vec{\tau}^{\mp,\mathrm{T}},s_0\vec{\tau}^{\mp}), \tag{36}$$

and $D_{\mathrm{J}}^{\pm}(t,\lambda)$ is the block-diagonal factor in (34),

$$D_{\mathrm{J}}^{+}=\begin{pmatrix} m_1^{+} & 0 & 0\\ 0 & \mathbf{m}_2^{+} & 0\\ 0 & 0 & 1/m_1^{+}\end{pmatrix},\qquad D_{\mathrm{J}}^{-}=\begin{pmatrix} 1/m_1^{-} & 0 & 0\\ 0 & \mathbf{m}_2^{-} & 0\\ 0 & 0 & m_1^{-}\end{pmatrix}. \tag{37}$$

Here, $m_k^{\pm}(t,\lambda)$ are the upper and lower rank k principal minors of the scattering matrix $T(t,\lambda)$ given by (34), and

$$\mathbf{m}_2^{+}=\mathbf{T}_{22}+\frac{\vec{b}^{+}\vec{B}^{-,\mathrm{T}}}{m_1^{+}},\qquad \mathbf{m}_2^{-}=\mathbf{T}_{22}+\frac{s_0\vec{B}^{-}\vec{b}^{+,\mathrm{T}}s_0}{m_1^{-}}. \tag{38}$$

If the potential matrix $Q(x,t)$ satisfies NLEE (9), then the associated scattering matrix $T(t,\lambda)$ evolves linearly in time, i.e., it satisfies the equation

$$i\,\frac{dT}{dt}+[f(\lambda),T(t,\lambda)]=0, \tag{39}$$

where $f(\lambda)$ is the dispersion law of NLEE (9). For NLS-type equations, we have $f(\lambda)=-\lambda^2 J$.

3.2. Fundamental analytic solutions. In this section, we briefly outline the construction of the FAS $\chi(x,t,\lambda)$ of the generalized ZS system (7) [39], [4]. The FAS can be directly obtained from the Jost solutions of (7),

$$\chi^{\pm}(x,t,\lambda)=\phi(x,t,\lambda)S_{\mathrm{J}}^{\pm}(t,\lambda)=\psi(x,t,\lambda)T_{\mathrm{J}}^{\mp}(t,\lambda)D_{\mathrm{J}}^{\pm}(t,\lambda), \tag{40}$$

by using generalized Gauss decomposition (34) of the scattering matrix $T(t,\lambda)$.

On the real axis (i.e., on the continuous spectrum of $L(\lambda)$), the two FAS are linearly dependent,

$$\chi^{+}(x,t,\lambda)=\chi^{-}(x,t,\lambda)G_0(t,\lambda),\quad \lambda\in\mathbb{R}, \tag{41}$$

where the sewing function $G_0(t,\lambda)$ can be expressed in terms of the generalized Gauss factors $S_{\mathrm{J}}^{\pm}(t,\lambda)$,

$$G_{0,J}(t,\lambda)=\widehat{S}_J^{-}(t,\lambda)S_J^{+}(t,\lambda)|_{t=0}. \tag{42}$$

The independent matrix elements of $G_0(t,\lambda)$ together with the discrete spectrum of $L(\lambda)$ form the minimum set of scattering data of L. Based on the completeness relations of the associated square solution and a Wronskian-type relation, we can recover the potential $Q(x,t)$ from the minimum set of scattering data [50].

We conclude this section by noting that although the general form of Gauss decomposition (34) holds for any symmetric space, we can simplify the form of the matrix block for symmetric spaces of type **A.III** by slightly modifying the Gauss decomposition of $T(\lambda)$:[1]

$$T(\lambda)=\mathbf{T}_J^{-}(\lambda)\widehat{\mathbf{S}}_J^{+}(\lambda)=\mathbf{T}_J^{+}(\lambda)\widehat{\mathbf{S}}_J^{-}(\lambda). \tag{43}$$

[1]Decompositions of type (43) are known as LU decompositions of $T(\lambda)$, while a decomposition of type (34) is known as a LDU decomposition of $T(\lambda)$.

Now $\mathbf{S}^{\pm}(\lambda)$ and $\mathbf{T}^{\pm}(\lambda)$ are the block-triangular matrices

$$\begin{aligned} \mathbf{S}_J^{+}(\lambda) &= \begin{pmatrix} \mathbb{1}_s, & \mathbf{d}^{-}(\lambda) \\ 0, & \mathbf{c}^{+}(\lambda) \end{pmatrix}, \qquad \mathbf{T}_J^{-}(\lambda) = \begin{pmatrix} \mathbf{a}^{+}(\lambda), & 0 \\ \mathbf{b}^{+}(\lambda), & \mathbb{1}_p \end{pmatrix}, \\ \mathbf{S}_J^{-}(\lambda) &= \begin{pmatrix} \mathbf{c}^{-}(\lambda), & 0 \\ -\mathbf{d}^{+}(\lambda), & \mathbb{1}_p \end{pmatrix}, \qquad \mathbf{T}_J^{+}(\lambda) = \begin{pmatrix} \mathbb{1}_s, & -\mathbf{b}^{-}(\lambda) \\ 0, & \mathbf{a}^{-}(\lambda) \end{pmatrix}. \end{aligned} \tag{44}$$

Using Gauss decomposition (43), we can write the sewing function $G_{0,J}(t,\lambda)$ given by (42) explicitly in terms of the blocks of $T(t,\lambda)$ and its inverse:

$$G_0(\lambda) = \widehat{D}^{-}(\lambda)(\mathbb{1} + K^{-}(\lambda)), \qquad \widehat{G}_0(\lambda) = \widehat{D}^{+}(\lambda)(\mathbb{1} - K^{+}(\lambda)). \tag{45}$$

Here, the block-diagonal factors $D^{\pm}(\lambda)$ and block-off-diagonal factors $K^{\pm}(\lambda)$ are respectively expressed in terms of the blocks $\mathbf{a}^{\pm}(\lambda)$ and $\mathbf{b}^{\pm}(\lambda)$ of $T(\lambda)$ given by (29) and in terms of the blocks $\mathbf{c}^{\pm}(\lambda)$ and $\mathbf{d}^{\pm}(\lambda)$ of its inverse $\widehat{T}(\lambda)$ given by (30):

$$\begin{aligned} D^{-}(\lambda) &= \begin{pmatrix} \mathbf{c}^{-}(\lambda) & 0 \\ 0 & \mathbf{a}^{-}(\lambda) \end{pmatrix}, \qquad K^{-}(\lambda) = \begin{pmatrix} 0 & \mathbf{d}^{-}(\lambda) \\ \mathbf{b}^{+}(\lambda) & 0 \end{pmatrix}, \\ D^{+}(\lambda) &= \begin{pmatrix} \mathbf{a}^{+}(\lambda) & 0 \\ 0 & \mathbf{c}^{+}(\lambda) \end{pmatrix}, \qquad K^{+}(\lambda) = \begin{pmatrix} 0 & \mathbf{b}^{-}(\lambda) \\ \mathbf{d}^{+}(\lambda) & 0 \end{pmatrix}. \end{aligned} \tag{46}$$

The superscripts $\pm$ in the expressions $D^{\pm}(\lambda)$ above mean analyticity for $\lambda \in \mathbb{C}_{\pm}$.

4. Dressing method and soliton solutions

4.1. The Zakharov–Shabat dressing method. The main idea of the ZS dressing method is to start with a regular solution $\eta_0^{\pm}(x,t,\lambda)$ of (7a) and "dress" it by adding singularities at two prescribed points $\lambda_k^{\pm}$ [3], [39], [40], [51]–[55]. The new singular solution has the form

$$\eta^{\pm}(x,t,\lambda) = u_k(x,t,\lambda)\eta_0^{\pm}(x,t,\lambda)w_{k,\pm}^{-1}(\lambda). \tag{47}$$

Here, the matrices $w_{k,\pm}(\lambda)$ can be given by

$$w_{k,+}(\lambda) = \mathbb{1}, \qquad w_{k,-}(\lambda) = \begin{pmatrix} u_{11,k}^{+} & 0 \\ 0 & u_{22,k}^{-} \end{pmatrix},$$

where the limits of the dressing factor can be obtained by

$$\begin{aligned} \lim_{x\to\infty} u_k(x,t,\lambda) &= \begin{pmatrix} u_{11,k}^{+} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad u_{11,k}^{+} = \mathbb{1} + (c_k(\lambda) - 1)P_{11,k}^{+}, \\ \lim_{x\to-\infty} u_k(x,t,\lambda) &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & u_{22,k}^{-} \end{pmatrix}, \qquad u_{22,k}^{-} = \mathbb{1} + (c_k(\lambda) - 1)P_{22,k}^{-}. \end{aligned} \tag{48}$$

For Lax operators related to **A.III**-symmetric spaces, the dressing factors have the forms

$$u_k(x,t,\lambda) = \mathbb{1} + (c_k(\lambda) - 1)P_k(x,t), \qquad c_k(\lambda) = \frac{\lambda - \lambda_k^{+}}{\lambda - \lambda_k^{-}}. \tag{49}$$

Here, $u_k(x,t,\lambda)$ is the dressing factor, and $P_k(x,t)$ is a projector of rank 1 with the form

$$P_k(x,t) = \frac{|n_k(x,t)\rangle\langle m_k(x,t)|}{\langle m_k(x,t)|n_k(x,t)\rangle}, \tag{50}$$

where $|n_k(x,t)\rangle = \chi_0^+(x,t,\lambda_1^+)|n_{0,1}\rangle$ is a column vector, $\langle m_1(x,t)| = \langle m_{0,1}|\widehat{\chi}_0^-(x,t,\lambda_1^-)$ is a row vector, and both of them are eigenvectors. The projectors $P_k(x,t)$ automatically satisfy the condition $P_k^2(x,t) = P_k(x,t)$. The functions $\eta^{\pm}(x,t,\lambda)$ satisfy the linear system

$$\left(i\,\frac{d\eta^{\pm}}{dx} + Q(x,t)\eta^{\pm}(x,t,\lambda) - \lambda[\mathrm{J},\eta^{\pm}(x,t,\lambda)]\right) = 0 \tag{51}$$

and are related to the ZS system with an unknown potential $Q(x,t)$, which can be found later, while $\eta_0^{\pm}(x,t,\lambda)$ is related to the ZS system with a known potential $Q_0(x,t)$:

$$i\,\frac{d\eta_0^{\pm}}{dx} + Q_0(x,t)\eta_0^{\pm}(x,t,\lambda) - \lambda[\mathrm{J},\eta_0^{\pm}(x,t,\lambda)] = 0. \tag{52}$$

From (47) and (52), we now find that the dressing factor $u_k(x,t,\lambda)$ satisfies the equation

$$i\,\frac{du_k}{dx} + Q(x,t)u_k(x,t,\lambda) - u_k(x,t,\lambda)Q_0(x,t) - \lambda[\mathrm{J},u_k(x,t,\lambda)] = 0. \tag{53}$$

Because the ansatz for the dressing factor $u_k(x,t,\lambda)$ in (49) and (53) is compatible with respect to λ, there are two conditions applicable to the left-hand side of (53), which are the limit as $\lambda\to\infty$ and the residue at $\lambda=\lambda_k^-$, and both of them vanish. The first condition leads to the potential

$$Q(x,t) - Q_0(x,t) = -(\lambda_k^+ - \lambda_k^-)[\mathrm{J},P_k(x,t)], \tag{54}$$

and the second condition gives the nonlinear equation for $P_k(x,t)$

$$\begin{aligned} i\,\frac{dP_k}{dx} &+ Q_0(x,t)P_k(x,t) - P_k(x,t)Q_0(x,t) - \lambda_k^+\mathrm{J}P_k(x,t) + \\ &+ \lambda_k^- P_k(x,t)\mathrm{J} + (\lambda_k^+ - \lambda_k^-)P_k(x,t)\mathrm{J}P_k(x) = 0. \end{aligned} \tag{55}$$

In addition, we have the normalization condition $\lim_{\lambda\to\infty} u_k(x,t,\lambda) = \mathbb{1}$.

In the case of **BD**.**I**-symmetric spaces, the dressing factor $u_k(x,t,\lambda)$ can be written in the form

$$u_k(x,t,\lambda) = \mathbb{1} + (c_k(\lambda)-1)P_k(x,t) + \left(\frac{1}{c_k(\lambda)} - 1\right)\overline{P}_k(x,t), \qquad \overline{P}_k = S_0P_k^{\mathrm{T}}S_0^{-1}, \tag{56}$$

where $P_k(x,t)$ and $\overline{P}_k(x,t)$ are mutually orthogonal projectors of rank 1 [21]. Similarly, we can write the dressed potential $Q(x,t)$ as

$$Q(x,t) = Q_0(x,t) - (\lambda_k^+ - \lambda_k^-)[\mathrm{J},P_k(x,t) - \overline{P}_k(x,t)]. \tag{57}$$

4.2. One-soliton solutions.

Example 3. If we take $\mathfrak{g} \simeq sl(3,\mathbb{C})$ and a dressing factor in form (49) satisfying nonlocal reduction conditions (16), then this implies the involution on the dressing factor

$$Bu_1(-x,t,-\lambda^*)^\dagger B^{-1} = u_1^{-1}(x,t,\lambda), \tag{58}$$

where B is a constant block-diagonal matrix. As a result, the projector P_1 must satisfy

$$P_1(x,t) = BP_1^\dagger(-x,t)B^{-1}, \qquad (-\lambda^\pm)^* = \lambda^\mp. \tag{59}$$

This means that the projector $P_1(x,t)$ and $c_1(\lambda)$ become

$$\begin{aligned} &P_1(x,t) = \frac{|n_1(x,t)\rangle\langle n_1^*(-x,t)|B}{\langle n_1^*(-x,t)|B|n_1(x,t)\rangle}, \qquad \langle m_1(x,t)| = (B|n_1(-x,t)\rangle)^\dagger, \\ &c_1(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda + (\lambda_1^+)^*}. \end{aligned} \tag{60}$$

From (54), we can therefore write the components of the one-soliton solution as

$$\begin{aligned} q_{1j}(x,t) &= -2(\lambda_1^+ + (\lambda_1^+)^*)\frac{n_{0,1}^1 (n_{0,1}^j)^* e^{-i(\widetilde{M}_1^+(x,t) + (\widetilde{M}_1^+)^*(-x,t))}}{2R_{0,1}\cosh(2\nu_1 x + 2\tilde{\mu}_1 t + \xi_{0,1})}, \\ q_{j1}(x,t) &= -2(\lambda_1^+ + (\lambda_1^+)^*)\frac{n_{0,1}^j (n_{0,1}^1)^* e^{i(\widetilde{M}_1^+(x,t) + (\widetilde{M}_1^+)^*(-x,t))}}{2R_{0,1}\cosh(2\nu_1 x + 2\tilde{\mu}_1 t + \xi_{0,1})}, \end{aligned} \qquad j = 2,3, \tag{61}$$

where

$$\begin{aligned} &\widetilde{M}_1^\pm(x,t) = \lambda_1^\pm x + 2f_{0,1}^\pm t, \qquad (\widetilde{M}_1^+(-x,t))^* = (-\lambda_1^+)^*(-x) + 2(f_0(-\lambda_1^+)^*)t, \\ &\nu_1 = \frac{i((-\lambda_1^+)^* - \lambda_1^+)}{2}, \qquad \tilde{\mu}_1 = i(f(-\lambda_1^+)^* - f(\lambda_1^+)), \\ &R_{0,1} = \sqrt{(n_{0,1}^1)^* n_{0,1}^1((n_{0,1}^2)^* n_{0,1}^2 + (n_{0,1}^3)^* n_{0,1}^3)}, \\ &\xi_{0,1} = \frac{1}{2}\log\frac{(n_{0,1}^1)^* n_{0,1}^1}{(n_{0,1}^2)^* n_{0,1}^2 + (n_{0,1}^3)^* n_{0,1}^3}. \end{aligned} \tag{62}$$

Example 4. If we again take $\mathfrak{g} \simeq sl(3,\mathbb{C})$ and the involution automorphism to be the Weyl reflection with respect to the second simple root $\mathrm{e}_2 - \mathrm{e}_3$ of $sl(3,\mathbb{C})$ and impose a reduction of type (B) in (16), then this corresponds to another block matrix $B = \left(\begin{smallmatrix} \mathbf{B}_+ & 0 \\ 0 & \mathbf{B}_- \end{smallmatrix}\right)$, where $\mathbf{B}_-$ is a block-off-diagonal matrix. The blocks can be written as

$$\mathbf{B}_+ = \mp 1, \qquad \mathbf{B}_- = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}. \tag{63}$$

The reduction conditions for the matrix blocks can be written as

$$\text{(C)}: \quad \begin{aligned} \mathbf{q}^-(x,t) &= -\mathbf{B}_-(\mathbf{q}^+(-x,t))^\dagger(\mathbf{B}_+)^{-1}, \\ \mathbf{q}^+(x,t) &= -\mathbf{B}_+(\mathbf{q}^-(-x,t))^\dagger(\mathbf{B}_-)^{-1}. \end{aligned} \tag{64}$$

If we now apply the involution of type (C) using (63), then the standard Manakov VNLS equation is reproduced as

$$\begin{aligned} &-iq_{1,t}+q_{1,xx}+2(q_1(x,t)q_2^*(-x,t)+q_2(x,t)q_1^*(-x,t))q_1(x,t)=0,\\ &-iq_{2,t}+q_{2,xx}+2(q_1(x,t)q_2^*(-x,t)+q_2(x,t)q_1^*(-x,t))q_2(x,t)=0. \end{aligned} \tag{65}$$

We can also write a one-soliton solution of the Manakov VNLS equation for this type of involution because dressing factor (49) is satisfied by involution (58) with (59). From (54), we can therefore write the components of the one-soliton solution as

$$\begin{aligned} q_{12}(x,t)&=-2(\lambda_1^+ +(\lambda_1^+)^*)n_{0,1}^1 e^{-i\widetilde{M}_1^+(x,t)}R_1^{-1}(x,t)(n_{0,1}^3)^* e^{-i\widetilde{M}_1^{+,*}(-x,t)},\\ q_{13}(x,t)&=-2(\lambda_1^+ +(\lambda_1^+)^*)n_{0,1}^1 e^{-i\widetilde{M}_1^+(x,t)}R_1^{-1}(x,t)(n_{0,1}^2)^* e^{-i\widetilde{M}_1^{+,*}(-x,t)},\\ q_{21}(x,t)&=-2(\lambda_1^+ +(\lambda_1^+)^*)n_{0,1}^2 e^{i\widetilde{M}_1^+(x,t)}R_1^{-1}(x,t)(n_{0,1}^1)^* e^{i\widetilde{M}_1^{+,*}(-x,t)},\\ q_{31}(x,t)&=-2(\lambda_1^+ +(\lambda_1^+)^*)n_{0,1}^3 e^{i\widetilde{M}_1^+(x,t)}R_1^{-1}(x,t)(n_{0,1}^1)^* e^{i\widetilde{M}_1^{+,*}(-x,t)}, \end{aligned} \tag{66}$$

where

$$\begin{aligned} &\widetilde{M}_1^\pm(x,t)=\lambda_1^\pm x+2f_{0,1}^\pm t, \qquad (\widetilde{M}_1^+(-x,t))^*=(-\lambda_1^+)^*(-x)+2(f_0(-\lambda_1^+)^*)t,\\ &R_1(x,t)=2R_{0,1}\cosh(2\nu_1 x+2\tilde{\mu}_1 t+\xi_{0,1}),\\ &\nu_1=\frac{i((-\lambda_1^+)^*-\lambda_1^+)}{2}, \qquad \tilde{\mu}_1=i(f(-\lambda_1^+)^*-f(\lambda_1^+)),\\ &R_{0,1}=\sqrt{(n_{0,1}^1)^* n_{0,1}^1((n_{0,1}^3)^* n_{0,1}^2+(n_{0,1}^2)^* n_{0,1}^3)},\\ &\xi_{0,1}=\frac{1}{2}\log\frac{(n_{0,1}^1)^* n_{0,1}^1}{(n_{0,1}^3)^* n_{0,1}^2+(n_{0,1}^2)^* n_{0,1}^3}. \end{aligned} \tag{67}$$

Example 5. If we take $\mathfrak{g}\simeq so(5,\mathbb{C})$, then the dressing factor takes form (56). If we impose nonlocal reduction conditions (16),

$$Bu_1(-x,t,-\lambda^*)^\dagger B^{-1}=u_1^{-1}(x,t,\lambda), \tag{68}$$

then this leads to the reduction condition

$$P_1(x,t)=BP_1^\dagger(-x,t)B^{-1}, \qquad (-\lambda^\pm)^*=\lambda^\mp. \tag{69}$$

This means that the projector $P_1(x,t)$ and $c_1(\lambda)$ become

$$\begin{aligned} &P_1(x,t)=\frac{|n_1(x,t)\rangle\,\langle n_1^*(-x,t)|B}{\langle n_1^*(-x,t)|B|n_1(x,t)\rangle}, \qquad \langle m_1(x,t)|=(B|n_1(-x,t)\rangle)^\dagger,\\ &c_1(\lambda)=\frac{\lambda-\lambda_1^+}{\lambda+(\lambda_1^+)^*}. \end{aligned} \tag{70}$$

Hence, taking the trivial solution $Q_0(x,t)=0$, we can write the components of the one-soliton solution as

$$\begin{aligned}
q_{12}^{+}(x,t) &= q_{45}^{+}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^2)^* e^{-iM_1^+(x,t)} - n_{0,1}^4(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}),\\
q_{13}^{+}(x,t) &= -q_{35}^{+}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^3)^* e^{-iM_1^+(x,t)} + n_{0,1}^3(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}),\\
q_{14}^{+}(x,t) &= q_{25}^{+}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^4)^* e^{-iM_1^+(x,t)} - n_{0,1}^2(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}),\\
q_{21}^{-}(x,t) &= q_{54}^{-}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^2(n_{0,1}^1)^* e^{i(M_1^-(-x,t))^*} - n_{0,1}^5(n_{0,1}^4)^* e^{iM_1^+(x,t)}),\\
q_{31}^{-}(x,t) &= -q_{53}^{-}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^3(n_{0,1}^1)^* e^{i(M_1^-(-x,t))^*} + n_{0,1}^5(n_{0,1}^3)^* e^{iM_1^+(x,t)}),\\
q_{41}^{-}(x,t) &= q_{52}^{-}(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^4(n_{0,1}^1)^* e^{i(M_1^+(-x,t))^*} - n_{0,1}^5(n_{01}^2)^* e^{iM_1^+(x,t)}),
\end{aligned} \tag{71}$$

where

$$\begin{aligned}
&(M_1^+(-x,t))^* = (-\lambda_1^+)^*(-x) + 2(f(-\lambda_1^+)^*)t,\\
&R_1(x,t) = 2R_{0,1}\cosh(2\nu_1 x + 2\tilde{\mu}_1 t + \xi_{0,1}) + S,\\
&\nu_1 = \frac{i((-\lambda_1^+)^* - \lambda_1^+)}{2}, \qquad \tilde{\mu}_1 = i(f(-\lambda_1^+)^* - f(\lambda_1^+)),\\
&S = (n_{0,1}^2)^* n_{0,1}^2 + (n_{0,1}^3)^* n_{0,1}^3 + (n_{0,1}^4)^* n_{0,1}^4,\\
&R_{0,1} = \sqrt{(n_{0,1}^1)^* n_{0,1}^1 + (n_{0,1}^5)^* n_{0,1}^5}, \qquad \xi_{0,1} = \frac{1}{2}\log\frac{(n_{0,1}^1)^* n_{0,1}^1}{(n_{0,1}^5)^* n_{0,1}^5}.
\end{aligned} \tag{72}$$

Example 6. If we again take $\mathfrak{g} \simeq so(5,\mathbb{C})$ and another $\mathbb{Z}_2$ reduction of type (B) given by (16) with a block matrix B containing an off-diagonal block $\mathbf{B}_\pm$, $\mathbf{B}_+ = \mathbf{B}_- = \mp 1$ and $\mathbf{B}_\pm = \mathrm{off} - \mathrm{diag}(\pm 1, \dots, \pm 1)$. In this case, the reductions conditions for the matrix blocks can be written as

$$\mathbf{q}^-(x,t) = -\mathbf{B}_-(\mathbf{q}^+(-x,t))^\dagger(\mathbf{B}_+)^{-1}, \qquad \mathbf{q}^+(x,t) = -\mathbf{B}_+(\mathbf{q}^-(-x,t))^\dagger(\mathbf{B}_-)^{-1}. \tag{73}$$

Recalling Example 1 in Sec. 2, if we now apply the involution of type (B) with $\mathbf{B}_+ = \mathbf{B}_- = \mp 1$ and $\mathbf{B}_\pm = \text{off-diag}(\pm 1, \dots, \pm 1)$, then we can rewrite the three-component NLS system in (24) as

$$\begin{aligned}
&i(q_t^+)_{12} + (q_{xx}^+)_{12} + 2(q_{12}(x,t)q_{14}^*(-x,t) +\\
&\qquad + 2q_{13}(x,t)q_{13}^*(-x,t))q_{12} + 2q_{12}^{+,*}(-x,t)(q_{13}^+)^2 = 0,\\
&i(q_t^+)_{13} + (q_{xx}^+)_{13} + 2(q_{12}(x,t)q_{14}^*(-x,t) + q_{13}(x,t)q_{13}^*(-x,t) +\\
&\qquad + q_{14}(x,t)q_{12}^*(-x,t))q_{13} + 2q_{13}^{+,*}(-x,t)q_{14}^+ q_{12}^+ = 0,
\end{aligned} \tag{74}$$

$$i(q_t^+)_{14} + (q_{xx}^+)_{14} + 2(q_{14}(x,t)q_{12}^*(-x,t) + \\ + 2q_{13}(x,t)q_{13}^*(-x,t))q_{14} + 2q_{14}^{+,*}(-x,t)(q_{13}^+)^2 = 0.$$

We can also write a one-soliton solution of this system for this type of involution because dressing factor (56) is satisfied by involution (68) with (69). From (57), we can therefore write the components of the one-soliton solution as

$$\begin{aligned}
q_{12}^+(x,t) &= q_{45}^+(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^4)^* e^{-iM_1^+(x,t)} - n_{0,1}^4(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}), \\
q_{13}^+(x,t) &= -q_{35}^+(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^3)^* e^{-iM_1^+(x,t)} + n_{0,1}^3(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}), \\
q_{14}^+(x,t) &= q_{25}^+(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^1(n_{0,1}^2)^* e^{-iM_1^+(x,t)} - n_{0,1}^2(n_{0,1}^5)^* e^{-i(M_1^+(-x,t))^*}), \\
q_{21}^-(x,t) &= q_{54}^-(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t)(n_{0,1}^2(n_{0,1}^1)^* e^{i(M_1^-(-x,t))^*} - n_{0,1}^5(n_{0,1}^2)^* e^{iM_1^+(x,t)}), \\
q_{31}^-(x,t) &= -q_{53}^-(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t) \times (n_{0,1}^3(n_{0,1}^1)^* e^{i(M_1^-(-x,t))^*} + n_{0,1}^5(n_{0,1}^3)^* e^{iM_1^+(x,t)}), \\
q_{41}^-(x,t) &= q_{52}^-(x,t) = \\
&= -(\lambda_1^+ + (\lambda_1^+)^*)R_1^{-1}(x,t) \times (n_{0,1}^4(n_{0,1}^1)^* e^{i(M_1^+(-x,t))^*} - n_{0,1}^5(n_{01}^4)^* e^{iM_1^+(x,t)}),
\end{aligned} \tag{75}$$

where

$$\begin{aligned}
&(M_1^+(-x,t))^* = (-\lambda_1^+)^*(-x) + 2(f(-\lambda_1^+)^*)t, \\
&R_1(x,t) = 2R_{0,1}\cosh(2\nu_1 x + 2\tilde{\mu}_1 t + \xi_{0,1}) + S, \\
&\nu_1 = \frac{i((-\lambda_1^+)^* - \lambda_1^+)}{2}, \qquad \tilde{\mu}_1 = i(f(-\lambda_1^+)^* - f(\lambda_1^+)), \\
&S = (n_{0,1}^4)^* n_{0,1}^2 + (n_{0,1}^3)^* n_{0,1}^3 + (n_{0,1}^2)^* n_{0,1}^4, \\
&R_{0,1} = \sqrt{(n_{0,1}^1)^* n_{0,1}^1 + (n_{0,1}^5)^* n_{0,1}^5}, \qquad \xi_{0,1} = \frac{1}{2}\log\frac{(n_{0,1}^1)^* n_{0,1}^1}{(n_{0,1}^5)^* n_{0,1}^5}.
\end{aligned} \tag{76}$$

4.3. Two-soliton solution.

Example 7. To obtain the two-soliton solution for $\mathfrak{g} \simeq sl(3,\mathbb{C})$, we can rewrite dressing factor (47) as

$$\eta^+(x,t,\lambda) = u_{1,2}(x,t,\lambda)\eta_0^+(x,t,\lambda)(\omega_{1,2}^{\pm}(\lambda))^{-1}, \tag{77a}$$

$$u_{1,2}(x,t,\lambda) = \mathbb{1} + (c_1(\lambda) - 1)P_1(x,t) + (c_2(\lambda) - 1)P_2(x,t). \tag{77b}$$

Therefore, the singular solution (two-soliton solution) with singularities located at $\lambda_1^{\pm}$ and $\lambda_2^{\pm}$ can be obtained as

$$Q(x,t) = Q_0(x,t) - \sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)[\mathrm{J}, P_k(x,t)]. \tag{78}$$

Assuming a canonical reduction of type (B) given by (16) with $B = \mathbb{I}$, we can write the components of the two-soliton solution from (78) as

$$q_{1j}(x,t) = -2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)\frac{n_{0,k}^k (n_{0,k}^j)^* e^{-i(\widetilde{M}_k^+(x,t)+(\widetilde{M}_k^+)^*(-x,t))}}{2R_{0,k}\cosh(2\nu_k x + 2\tilde{\mu}_k t + \xi_{0,k})},$$
$$q_{j1}(x,t) = 2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)\frac{n_{0,k}^j (n_{0,k}^1)^* e^{i(\widetilde{M}_k^+(x,t)+(\widetilde{M}_k^+)^*(-x,t))}}{2R_{0,k}\cosh(2\nu_k x + 2\tilde{\mu}_k t + \xi_{0,k})}, \qquad j = 2,3, \tag{79}$$

where

$$\widetilde{M}_k^{\pm}(x,t) = \lambda_k^{\pm}x + 2f_{0,k}^{\pm}t, \qquad (\widetilde{M}_k^+(-x,t))^* = (-\lambda_k^+)^*(-x) + 2(f(-\lambda_1^+)^*)t,$$
$$\nu_k = \frac{i((-\lambda_k^+)^* - \lambda_k^+)}{2}, \qquad \tilde{\mu}_k = i(f(-\lambda_k^+)^* - f(\lambda_k^+)),$$
$$R_{0,k} = \sqrt{(n_{0,k}^1)^* n_{0,k}^1((n_{0,k}^2)^* n_{0,k}^2 + (n_{0,k}^3)^* n_{0,k}^3)}, \tag{80}$$
$$\xi_{0,k} = \frac{1}{2}\log\frac{(n_{0,k}^1)^* n_{0,k}^1}{(n_{0,k}^2)^* n_{0,k}^2 + (n_{0,k}^3)^* n_{0,k}^3}.$$

Remark 1. Alternatively, to obtain a two-soliton solution, we can again apply the dressing method to the one-soliton solution used as a seed solution:

$$\eta^+(x,t,\lambda) = u_2(x,t,\lambda)\eta_1^+(x,t,\lambda)(\omega_2^{\pm}(\lambda))^{-1}, \qquad u_2(x,t,\lambda) = \mathbb{1} + (c_2(\lambda)-1)P_2(x,t). \tag{81}$$

Example 8. In the case of the involution in Example 4, two-soliton dressing factor (77a) is automatically compatible with the involution. Therefore, we can write the components of the two-soliton solution from (78) as

$$q_{12}(x,t) = -2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)n_{0,k}^1 e^{-i\widetilde{M}_k^+(x,t)}R_k^{-1}(x,t)(n_{0,k}^3)^* e^{-i\widetilde{M}_k^{+,*}(-x,t)},$$
$$q_{13}(x,t) = -2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)n_{0,k}^1 e^{-i\widetilde{M}_k^+(x,t)}R_k^{-1}(x,t)(n_{0,k}^2)^* e^{-i\widetilde{M}_k^{+,*}(-x,t)},$$
$$q_{21}(x,t) = -2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)n_{0,k}^2 e^{i\widetilde{M}_k^+(x,t)}R_k^{-1}(x,t)(n_{0,k}^1)^* e^{i\widetilde{M}_k^{+,*}(-x,t)}, \tag{82}$$
$$q_{31}(x,t) = -2\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)n_{0,k}^3 e^{i\widetilde{M}_k^+(x,t)}R_k^{-1}(x,t)(n_{0,k}^1)^* e^{i\widetilde{M}_k^{+,*}(-x,t)},$$

where

$$\widetilde{M}_k^{\pm}(x,t) = \lambda_k^{\pm}x + 2f_{0,k}^{\pm}t, \qquad (\widetilde{M}_k^+(-x,t))^* = (-\lambda_k^+)^*(-x) + 2(f(-\lambda_1^+)^*)t,$$
$$R_1(x,t) = 2R_{0,1}\cosh(2\nu_1 x + 2\tilde{\mu}_1 t + \xi_{0,1}),$$
$$\nu_k = \frac{i((-\lambda_k^+)^* - \lambda_k^+)}{2}, \qquad \tilde{\mu}_k = i(f(-\lambda_k^+)^* - f(\lambda_k^+)), \tag{83}$$
$$R_{0,k} = \sqrt{(n_{0,k}^1)^* n_{0,k}^1((n_{0,k}^3)^* n_{0,k}^2 + (n_{0,k}^2)^* n_{0,k}^3)},$$
$$\xi_{0,k} = \frac{1}{2}\log\frac{(n_{0,k}^1)^* n_{0,k}^1}{(n_{0,k}^3)^* n_{0,k}^2 + (n_{0,k}^2)^* n_{0,k}^3}.$$

Example 9. If we take $\mathfrak{g} \simeq so(5,\mathbb{C})$, then we can consider the two-soliton dressing factors $u(x,t,\lambda)$ with two more poles in the form

$$u_{1,2}(x,t,\lambda) = \mathbb{1} + (c_1(\lambda)-1)P_1(x,t) + \left(\frac{1}{c_1(\lambda)} - 1\right)\overline{P}_1(x,t) + \\ + (c_2(\lambda)-1)P_2(x,t) + \left(\frac{1}{c_2(\lambda)} - 1\right)\overline{P}_2(x,t), \tag{84}$$

$$c_k(\lambda) = \frac{\lambda-\lambda_k^+}{\lambda-\lambda_k^-}, \qquad P_k(x,t) = \frac{|n_k(x,t)\rangle\langle m_k(x,t)|}{\langle m_k(x,t)|n_k(x,t)\rangle}, \qquad \overline{P}_k = S_0 P_k^{\mathrm{T}} S_0^{-1}, \quad k=1,2. \tag{85}$$

We can hence obtain the new potential $Q(x,t)$ corresponding to a given trivial solution $Q_0(x,t)=0$ in the form

$$Q(x,t) = -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)[\mathrm{J}, P_k(x,t) - \overline{P}_k(x,t)]. \tag{86}$$

We can obtain the components of the new potential $Q(x,t)$ as

$$\begin{aligned}
q_{12}^+(x,t) &= q_{45}^+(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(n_{0,k}^1 m_{0,k}^2 e^{-iM_k^+(x,t)} + n_{0,k}^4 m_{0,k}^5 e^{-iM_k^-(x,t)}),\\
q_{13}^+(x,t) &= -q_{35}^+(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(n_{0,k}^1 m_{0,k}^3 e^{-iM_k^+(x,t)} - n_{0,k}^3 m_{0,k}^5 e^{-iM_k^-(x,t)}),\\
q_{14}^+(x,t) &= q_{25}^+(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(n_{0,k}^1 m_{0,k}^4 e^{-iM_k^+(x,t)} + n_{0,k}^2 m_{0,k}^5 e^{-iM_k^-(x,t)}),\\
q_{21}^-(x,t) &= q_{54}^-(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(-n_{0,k}^2 m_{0,k}^1 e^{iM_k^-(x,t)} - n_{0,k}^5 m_{0,k}^4 e^{iM_k^+(x,t)}),\\
q_{31}^-(x,t) &= -q_{53}^-(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(-n_{0,k}^3 m_{0,k}^1 e^{iM_k^-(x,t)} + n_{0,k}^5 m_{0,k}^3 e^{iM_k^+(x,t)}),\\
q_{41}^-(x,t) &= q_{52}^-(x,t) = \\
&= -\sum_{k=1}^{2}(\lambda_k^+ - \lambda_k^-)R_k^{-1}(x,t)(-n_{0,k}^4 m_{0,k}^1 e^{iM_k^-(x,t)} - n_{0,k}^5 m_{0,k}^2 e^{iM_k^+(x,t)}),
\end{aligned} \tag{87}$$

where

$$
\begin{aligned}
&M_k^{\pm}(x,t) = \lambda_k^{\pm}x + 2f_{0,k}^{\pm}t, \qquad R_k(x,t) = 2R_{0,k}\cosh(2\nu_k x + 2\tilde{\mu}_k t + \xi_{0,k}) + S,\\
&\nu_k = \frac{i(\lambda_k^- - \lambda_k^+)}{2}, \qquad \tilde{\mu}_k = i(f(\lambda_k^-) - f(\lambda_k^+)),\\
&S = m_{0,k}^2 n_{0,k}^2 + m_{0,k}^3 n_{0,k}^3 + m_{0,k}^4 n_{0,k}^4,\\
&R_{0,k} = \sqrt{m_{0,k}^1 n_{0,k}^1 + m_{0,k}^5 n_{0,k}^5}, \qquad \xi_{0,k} = \frac{1}{2}\log\frac{m_{0,k}^1 n_{0,k}^1}{m_{0,k}^5 n_{0,k}^5}.
\end{aligned}
\tag{88}
$$

Example 10. Again in the case where $\mathfrak{g} \simeq so(5,\mathbb{C})$, if we impose a nonlocal involution of form (16), then the dressing factor in (84) satisfies the reduction condition

$$
B\prod_{k=1}^{2} u_k(-x,t,-\lambda^*)^{\dagger}B^{-1} = \prod_{k=1}^{2} u_k^{-1}(x,t,\lambda),
\tag{89}
$$

where B is a constant block-diagonal matrix. This involution is satisfied if

$$
P_k(x,t) = BP_k^{\dagger}(-x,t)B^{-1}, \qquad (-\lambda^{\pm})^* = \lambda^{\mp}.
$$

This means that the projector $P_k(x,t)$ and $c_k(\lambda)$ become

$$
\begin{aligned}
&P_k(x,t) = \frac{|n_k(x,t)\rangle\langle n_k^*(-x,t)|B}{\langle n_k^*(-x,t)|B|n_k(x,t)\rangle}, \qquad \langle m_k(x,t)| = (B|n_k(-x,t)\rangle)^{\dagger},\\
&c_k(\lambda) = \frac{\lambda - \lambda_k^+}{\lambda + (\lambda_k^+)^*}.
\end{aligned}
\tag{90}
$$

It hence follows from (57) that

$$
Q(x,t) = Q_0(x,t) - (\lambda_k^+ + (\lambda_k^+)^*)[\mathrm{J}, BP_k^{\dagger}(-x,t)B^{-1}].
\tag{91}
$$

We can therefore write the components of the two-soliton solution as

$$
\begin{aligned}
q_{12}^+(x,t) &= q_{45}^+(x,t) =\\
&= -\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)R_k^{-1}(x,t)(n_{0,k}^1(n_{0,k}^2)^* e^{-iM_k^+(x,t)} - n_{0,k}^4(n_{0,k}^5)^* e^{-i(M_k^+(-x,t))^*}),\\
q_{13}^+(x,t) &= -q_{35}^+(x,t) =\\
&= -\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)R_k^{-1}(x,t)(n_{0,k}^1(n_{0,k}^3)^* e^{-iM_k^+(x,t)} + n_{0,k}^3(n_{0,k}^5)^* e^{-i(M_k^+(-x,t))^*}),\\
q_{14}^+(x,t) &= q_{25}^+(x,t) =\\
&= -\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)R_k^{-1}(x,t)(n_{0,k}^1(n_{0,k}^4)^* e^{-iM_k^+(x,t)} - n_{0,k}^2(n_{0,k}^5)^* e^{-i(M_k^+(-x,t))^*}),\\
q_{21}^-(x,t) &= q_{54}^-(x,t) =\\
&= -\sum_{k=1}^{2}(\lambda_k^+ + (\lambda_k^+)^*)R_k^{-1}(x,t)(n_{0,k}^2(n_{0,k}^1)^* e^{i(M_k^-(-x,t))^*} - n_{0,k}^5(n_{0,k}^4)^* e^{iM_k^+(x,t)}),
\end{aligned}
\tag{92}
$$

$$q_{31}^{-}(x,t) = -q_{53}^{-}(x,t) =$$

$$= -\sum_{k=1}^{2} (\lambda_k^+ + (\lambda_k^+)^*) R_k^{-1}(x,t) (n_{0,k}^3 (n_{0,k}^1)^* e^{i(M_k^-(-x,t))^*} + n_{0,k}^5 (n_{0,k}^3)^* e^{iM_k^+(x,t)}),$$

$$q_{41}^{-}(x,t) = q_{52}^{-}(x,t) =$$

$$= -\sum_{k=1}^{2} (\lambda_k^+ + (\lambda_k^+)^*) R_k^{-1}(x,t) (n_{0,k}^4 (n_{0,k}^1)^* e^{i(M_k^+(-x,t))^*} - n_{0,k}^5 (n_{0,k}^2)^* e^{iM_k^+(x,t)}),$$

where

$$\begin{aligned}
&(M_k^+(-x,t))^* = (-\lambda_k^+)^*(-x) + 2(f(-\lambda_1^+)^*)t, \\
&R_k(x,t) = 2R_{0,k}\cosh(2\nu_k x + 2\tilde{\mu}_k t + \xi_{0,k}) + S, \\
&\nu_k = \frac{i((-\lambda_k^+)^* - \lambda_k^+)}{2}, \qquad \tilde{\mu}_k = i(f(-\lambda_k^+)^* - f(\lambda_k^+)), \\
&S = (n_{0,k}^2)^* n_{0,k}^2 + (n_{0,k}^3)^* n_{0,k}^3 + (n_{0,k}^4)^* n_{0,k}^4, \\
&R_{0,k} = \sqrt{(n_{0,k}^1)^* n_{0,k}^1 + (n_{0,k}^5)^* n_{0,k}^5}, \qquad \xi_{0,k} = \frac{1}{2}\log\frac{(n_{0,k}^1)^* n_{0,k}^1}{(n_{0,k}^5)^* n_{0,k}^5}.
\end{aligned} \tag{93}$$

5. Conclusions

We have studied multicomponent generalizations of NLS models on **A**.**III**- and **BD**.**I**-symmetric spaces. Our study was based on two types of main examples: the Manakov VNLS equation, related to symmetric spaces of **A**.**III** type, and KS models, related to symmetric spaces of **BD**.**I** type. We first formulated the direct scattering problem for both models, including the construction of the Jost solutions, the scattering matrix, and the minimum set of scattering data. Based on the Gauss decomposition of the scattering matrix, we also constructed the FAS.

It turns out that the spectral properties of the Lax operator depend crucially on the choice of the representation of the underlying Lie algebra or symmetric space while the minimum set of scattering data is provided by the same set of functions [43].

Finally, we presented a modification of the dressing method and obtained one- and two-soliton solutions for both models with nonlocal reductions. Depending on the positions of the discrete eigenvalues $\lambda_j^{\pm}$ in the spectral plane, there are two regimes for the two-soliton solution: if two of the discrete eigenvalues are in the upper half of the complex plane while the other two are in the lower half, then the nonlocal involution preserves their number inside each of the contours, we hence have a Riemann–Hilbert problem with a balanced number of singularities, and the corresponding two-soliton solutions are therefore regular for all t. Otherwise, the two-soliton solution again develops a singularity in a finite time as in the case studied in [23]–[25].

The results in this paper can be extended in several directions:

1. To construct a gauge covariant formulation of the MNLS hierarchies on symmetric spaces, including the Wronskian relations, the "squared solutions," and their completeness relations, the descriptions of the class of NLEEs associated with a given scattering problem, the generating (recursion) operator and its spectral decomposition, the description of the infinite set of integrals of motion, the hierarchy of Hamiltonian structures, and the r-matrix formulation.

2. To investigate gauge-equivalent systems of multicomponent ferromagnetics on symmetric spaces [52], [56], [57].

3. To study different types of reductions of multicomponent integrable systems on symmetric spaces [58]–[60].

4. To study the associated Darboux transformations and their generalizations for NLS equations over Hermitian symmetric spaces and to obtain multisoliton solutions via such generalizations, which also includes rational solutions [55].

5. To extend our results to the case where the boundary conditions are nonvanishing (a nontrivial background) [61], [62] (the considerations required in this case are more complicated and will be discussed it elsewhere).

6. To study other types of integrable hierarchies on symmetric spaces, for example, quadratic bundles (related to the derivative NLS equation, the Kaup–Newell equation, or the Gerdjikov–Ivanov equation) [4], [63]–[67] or rational bundles [56], [57].

Acknowledgments. The authors are pleased to thank Professors Mark Ablowitz and Vladimir Gerdjikov for the numerous useful discussions.

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