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Quantum dynamics of a parity-time-symmetric kicked particle in a 1D box

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Abstract

We study quantum particle dynamics in a box and driven by a parity-time (PT)-symmetric, delta-kicking complex potential. The dynamical characteristics, identified by the average kinetic energy as a function of time, and quasi-energy are computed for different values of the kicking parameters. Breaking of the PT-symmetry at certain values of the non-Hermitian kicking parameter is shown. Experimental realization of the model is also discussed.

Keywords: PT-symmetry, delta-kicked particle in a box, quantum dynamics

(Some figures may appear in colour only in the online journal)

1. Introduction

Parity-time (PT)-symmetric quantum systems attracted much attention during past two decades after the discovery of the fact that non-Hermitian, but PT-symmetric, systems can have a set of eigenstates with real eigenvalues [1]. In other words, self-adjointness of the Hamiltonian is not a necessary condition for the eigenvalues to be real. Currently quantum physics of PT-symmetric systems has become a rapidly developing topic of contemporary physics and a great progress has been made in the study of different aspects of such systems (see, e.g. papers [2–24] for reviews of recent developments on the topic). These studies allowed us to construct a complete theory of PT-symmetric quantum systems, including PT-symmetric field theory [8, 15]. Experimental realization of PT-symmetric systems was mainly done in optics [25, 26, 29, 32]. However, some other versions of PT-symmetric systems were discussed recently in the literature [34–39]. PT-symmetric relativistic systems were also studied in [19–21]. A general condition for PT-symmetry in quantum systems has been derived in terms of the so-called charge-parity-time (CPT)-symmetric inner product [5, 10, 15]. PT-symmetry in quantum systems can be introduced either through the complex potential or by imposing proper boundary conditions, which provide the symmetry via the CPT-inner product [5, 15]. Different types of complex potentials providing PT-symmetry in Hamiltonian have been considered in [10, 15, 34, 36]. PT-symmetric particle-in-a-box systems, where the box boundary conditions provide PT-symmetry of the system, have been studied in [14, 22–24]. A certain progress has also been done in nonlinear extensions of PT-symmetric systems [27–32].

In this paper we consider a quantum particle confined in a 1D box and driven by a PT-symmetric, delta-kicking potential with the focus on acceleration and PT-symmetry breaking. Classical and quantum dynamics of systems interacting with a delta-kicking potential have been extensively studied in the context of nonlinear dynamics and quantum chaos theory [40–46]. Kicked quantum particle dynamics in a box have been also considered in [47–50]. For kicked systems, the classical dynamics is characterized by a diffusive growth of the average kinetic energy as a function of time, while for the corresponding quantum systems such a growth is suppressed (except in the special cases of the so-called quantum resonances). The latter is called quantum localization of classical chaos [40–46]. The dynamics of kicked nonrelativistic systems is governed by a single parameter, which is the product of the kicking strength and kicking period.

We note that earlier, classical dynamics of PT-symmetric kicked systems was considered in [33], while corresponding quantum systems driven by PT-symmetric delta kicks have been considered in [34, 36] in the context of quantum chaos theory. In [34] a PT-symmetric kicked rotor is studied by developing a one-parameter scaling theory for the non-Hermitian parameter and focusing on the gain-loss effects. In [36] a PT-symmetric kicked quantum rotor is studied by analyzing the quasienergy spectrum and evolution of the momentum distribution at different values of the non-Hermitian parameter. Here we consider a PT-symmetric kicked confined system.

We note that PT-symmetric kicked quantum box can be realized in different versions. The usual way of creating a kicked quantum system is confining the system in a standing wave cavity. A PT-symmetric analog of such a system could be realized in a cavity with the losses. Another option, putting the system in a transverse beam propagation inside a passive optical resonator with combined phase and loss gratings, was discussed, e.g. in [36]. Periodic array of optical waveguides driven by PT-symmetric optical field can be also considered within the model to be treated below.

This paper is organized as follows. In the next section we briefly recall Hermitian counterpart of our system, quantum particle confined in a 1D box and driven by a delta-kicking potential. In section 3 we consider a similar system with PT-symmetric delta-kicks. Section 4 presents some concluding remarks.

2. Kicked quantum particle dynamics in a box

The Hermitian counterpart of the system we are going to study is a quantum particle confined in a one-dimensional box of size L and driven by an external delta-kicking potential, which is given by

$$U(x,t) = V(x)\sum_{l}\delta(t-lT),$$

with $V(x) = \epsilon\cos(2\pi x/\mu)$ and where μ, ϵ and T are the wavelength, kicking strength and kicking period, respectively. Such systems were considered earlier in the context of quantum localization and chaos theory, e.g. in [47–50] and described by the following time-dependent Schrödinger equation (in the units $\hbar = m = 1$):

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x,t)\right]\Psi(x,t). \tag{1}$$

The wave function, $\Psi(x,t)$ fulfills the box boundary conditions given by

$$\Psi(0,t) = \Psi(L,t) = 0. \tag{2}$$

Exact solutions of equation (1) can be obtained within the single kicking period [40, 48] by expanding the wave function, $\Psi(x,t)$ in terms of the complete set of the eigenfunctions of the unperturbed system as

$$\Psi(x,t) = \sum_n A_n(t)\psi_n(x), \tag{3}$$

where $\psi_n(x) = \sqrt{2/L}\sin(\pi nx/L)$. Equations (1) and (3) lead to a quantum mapping for the wave function amplitudes $A_n(t)$, which is given by

$$A_n(t+T) = \sum_l A_l(t)U_{ln}\mathrm{e}^{-\mathrm{i}E_lT}, \tag{4}$$

where

$$U_{ln} = \int_0^L \psi_n^*(x)\mathrm{e}^{-\mathrm{i}V(x)}\psi_l(x)\mathrm{d}x$$

and

$$E_l = (\pi l/L)^2. \tag{5}$$

The amplitudes fulfill the norm conservation given by

$$N(t) = \sum_n |A_n(t)|^2 = 1. \tag{6}$$

In the following we use this condition to control the accuracy of our numerical computations. Thus the evolution of the wave function within the single kicking period can be written as

$$\Psi(x,t+T) = \hat{U}\Psi(x,t),$$

where the one-period evolution operator is given by

$$\hat{U} = \exp\left(-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial x^2}\right)\exp\left(-\mathrm{i}\beta V(x)\right)\exp\left(-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial x^2}\right), \tag{7}$$

where $\beta = \pi T/\mu^2$.

For such an operator, one can consider the eigenvalue problem given by

$$\hat{U}\phi_n = \lambda_n\phi_n, \tag{8}$$

where the eigenvalues, λ_n are called quasienergy levels of the kicked system. In figure 1 few quasienergy levels are plotted as a function of the wave number, $k = 2\pi/\mu$ at different values of the parameter $K = \epsilon T$. The higher the value of K, the stronger the fluctuations of the quasienergy levels. All the curves have a maximum at $k = 0$, which is caused by the fact that the kicking potential, containing the factor $\cos kx$, reaches its maximum value at $k = 0$

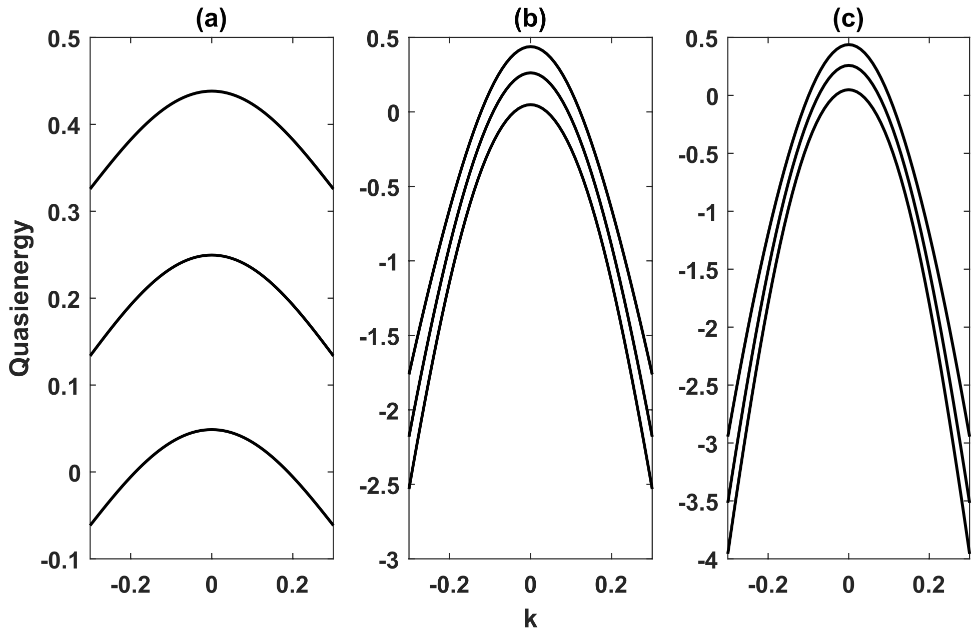

Figure 1. Few quasienergy levels, determined by equation (8), as a function of the wave number $k = 2\pi/\mu$, for $L = 10$ and different $K = \epsilon T$: $K = 0.05$ (a), $K = 1$ (b), $K = 1.5$ (c).

$(\cos kx = 1)$. Plotting these curves for a wider interval in k shows that λ_n is quasiperiodic in k, which is also caused by the periodic of the kicking potential on k.

Having found amplitudes and wave function, one can compute the average kinetic energy, which is defined as

$$\begin{aligned} \langle E_k(t)\rangle &= -\frac{1}{2}\langle\Psi(x,t)|\frac{\mathrm{d}^2}{\mathrm{d}x^2}|\Psi(x,t)\rangle \\ &= \sum_n E_n|A_n(t)|^2, \end{aligned} \tag{9}$$

where E_n are given by equation (5). Figure 2 presents plots of the average kinetic energy, $\langle E_{\mathrm{k}}(t)\rangle$ at different values of the kicking strength, ε for fixed kicking period T. Unlike the kicked rotor, $\langle E_{\mathrm{k}}(t)\rangle$ grows during some initial time and suppression with the subsequent decrease occurs for a large enough number of kicks ($N = t/T$). For a very large number of kicks, one can observe a periodic or quasi-periodic time-dependence of $\langle E_{\mathrm{k}}(t)\rangle$. Such behaviors in some kicked quantum systems have been discussed earlier in [43]. The stronger the kick, the quicker the growth of $\langle E_{\mathrm{k}}(t)\rangle$ and the larger its amplitude. Another feature of a kicked quantum particle confined in a box is the absence of quantum resonances. It should be noted that the dynamics of a kicked particle confined in a box depends on two factors, namely the interaction with the kicking force and bouncing of particle from the box walls. Depending on the sign of cosine in the kicking potential, the kicking force can be attractive and repulsive. When the kicking potential is repulsive, particle gains energy, while in case of attractive potential it losses energy. Therefore depending on the localization area of the motion, acceleration or deceleration of the particle may occur. A very important factor is 'synchronization'

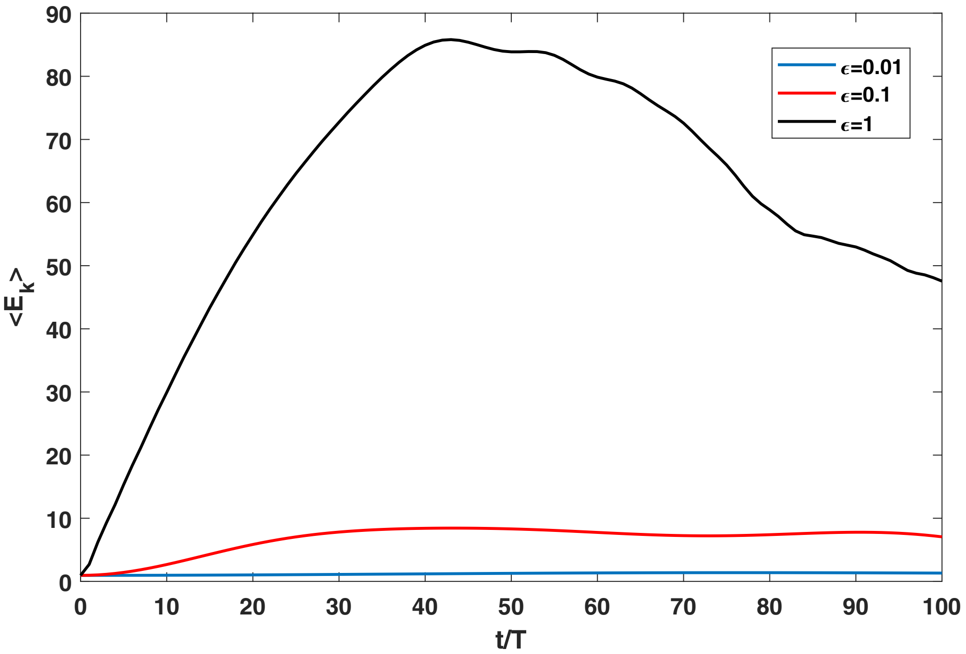

Figure 2. The average kinetic energy of kicked particle in a box as a function of kick number for different kicking strength for $L = 3.3$, $T = 0.01$ and $\mu = 1.3$.

of the kicking force and bouncing of particle from the box wall. It also may cause acceleration and deceleration of the particle.

3. PT-symmetric kicked quantum particle a one dimensional box

A PT-symmetric analog of the above system can be constructed by adding into the kicking potential an imaginary part. Then the PT-symmetric kicking potential can be written as

$$U_{\mathrm{PT}}(x,t) = V(x)\sum_{l}\delta(t-lT), \tag{10}$$

with $V(x) = [\epsilon\cos(2\pi x/\mu) + \mathrm{i}\gamma\sin(2\pi x/\mu)]$ and where ϵ and T are the kicking strength and period, respectively, $\gamma \geqslant 0$ is the non-Hermitian parameter that measures the strength of the imaginary part of the kick. The dynamics of the system is governed by the following time-dependent Schrödinger equation:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(x,t) = H_{\mathrm{PT}}\Psi(x,t), \tag{11}$$

where H_{PT} is the Schrödinger operator containing potential U_{PT}. The same boundary conditions as those in equation (2) are imposed for the wave function. Exact solutions of equation (11) can be obtained similarly to the case of the Hermitian counterpart and one gets a quantum mapping for the evolution of the amplitude, $A_n(t)$ within the one kicking period, T:

$$A_n(t+T) = \sum_{l} A_l(t)V_{ln}\mathrm{e}^{-\mathrm{i}E_lT}. \tag{12}$$

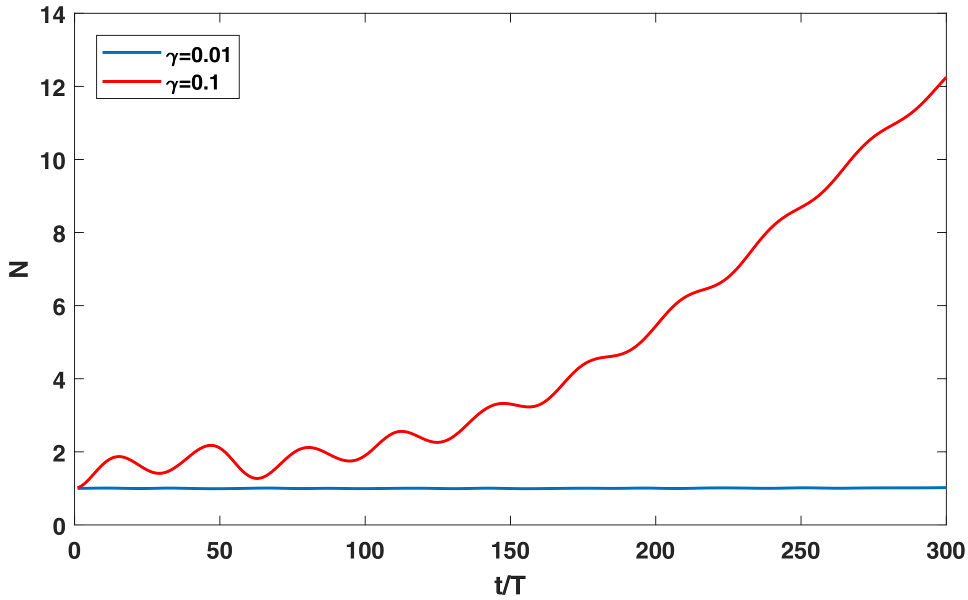

Figure 3. The norm as a function of kick number at different values of γ for $\epsilon = 0.1$, $L = 3.3$, $T = 0.01$ and $\mu = 1.3$.

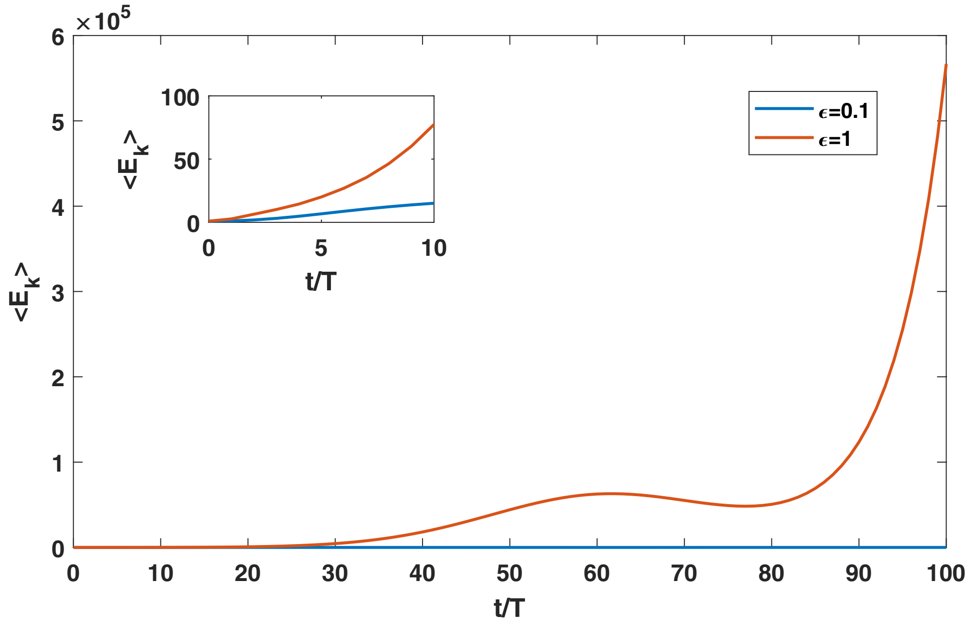

Figure 4. The average kinetic energy as a function of kick number for different kicking strengths for $\gamma = 0.1$, $L = 3.3$, $T = 0.01$ and $\mu = 1.3$.

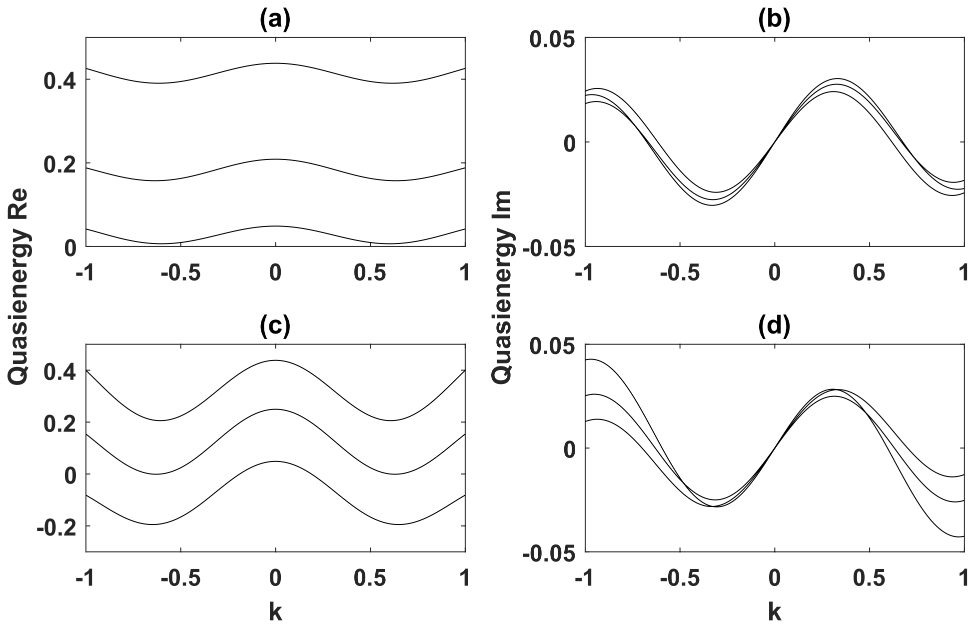

Figure 5. The real (a) and (c) and imaginary (b) and (d) parts of few quasienergy levels as a function of the wave number, $k = 2\pi/\mu$, for $\gamma = 1$ and for $\epsilon = 1$ (a) and (b) and $\epsilon = 5$ (c) and (d) for $L = 10$.

Here,

$$V_{ln} = \int \psi_n^*(x) \mathrm{e}^{-\mathrm{i}\epsilon\cos(2\pi x/\mu)} e^{\gamma\sin(2\pi x/\mu)} \psi_l(x) \mathrm{d}x, \tag{13}$$

and $E_l = (\pi l/L)^2$. The evolution operator corresponding to equation (12) can be written as

$$\hat{U}_{\mathrm{PT}} = \exp\left(-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial x^2}\right) \exp(-\mathrm{i}\beta V(x)) \exp\left(-\frac{\mathrm{i}}{2}\frac{\partial^2}{\partial x^2}\right), \tag{14}$$

where $\beta = \pi T/\mu^2$.

For quantum systems with complex PT-symmetric potentials, the norm conservation is broken, i.e. the amplitudes, $A_n(t)$ do not fulfill equation (6) that can be seen from figure 3, where the plots of the norm as a function of time at different values of γ for fixed ϵ and T are presented. The higher the value of γ, the stronger the breaking of the norm conservation. In figure 4 the average kinetic energy is plotted as a function of time. Although the plot profile is almost similar to that of the Hermitian counterpart, the values of $\langle E_{\mathrm{k}}(t)\rangle$ are much higher than that in the Hermitian case. Similarly to the Hermitian case, one can compute quasienergy levels for PT-symmetric system as the eigenvalues of the operator U_{PT}. In figure 5 real and imaginary parts of few quasienergy levels are plotted as a function of the wave number, $k = 2\pi/\mu$ at different values of the parameter ϵ. The behavior of the real part is similar to that of the Hermitian counterpart and the curves have maximum at $k = 0$, where $\cos kx$ reaches its maximum value. The curves are quasiperiodic in k that is also caused by the periodicity of the kicking potential on k.

Figure 6 presents the average total energy (a) and quasienergy (b) as a function of the parameter γ. An important feature of PT-symmetric systems with time-independent complex potentials is the fact that the expectation value of the Hamiltonian operator (energy levels) is always real. However, for time-dependent PT-symmetric potentials the situation is completely

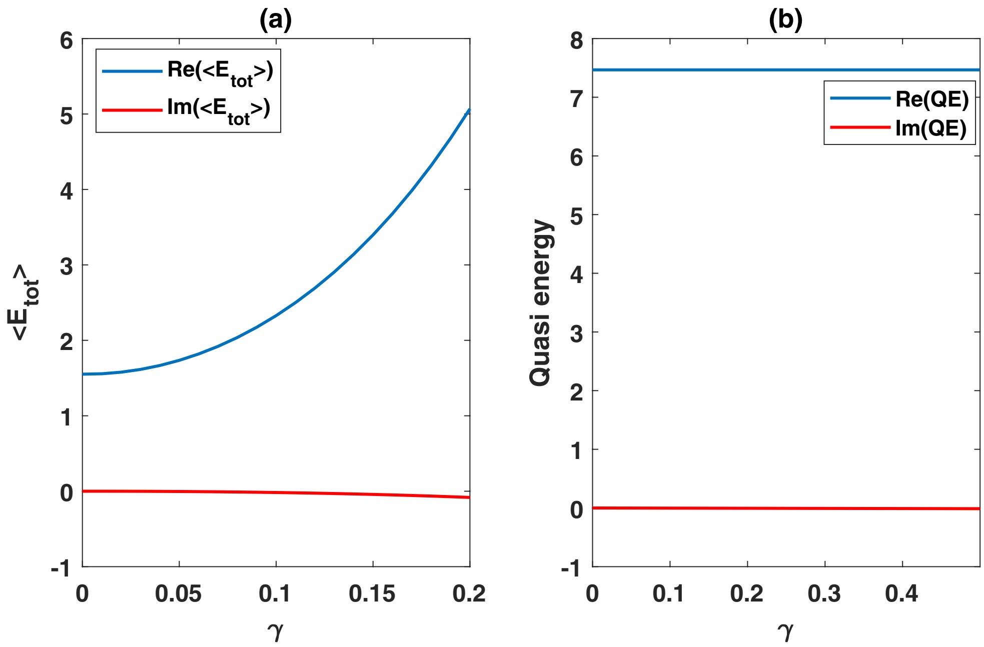

Figure 6. The average total energy (a) and quasienergy (b) as a function of γ for $\epsilon = 0.1$, $T = 0.01$, $\mu = 1.3$, $L = 3.3$ (a) and $\epsilon = 0.1$, $T = 0.01$, $k = 1$, $L = 10$ (b).

different. As shown in the recent studies of PT-symmetric kicked rotors in [34, 36], the average quasienergy becomes complex starting from some (critical) values of the non-Hermitian parameter, γ. This effect was called PT-symmetry breaking [34, 36]. Here, we check such effects for our system and found that the imaginary parts of the average total energy and average quasienergy become non-zero starting from some values of γ. In figure 6(a), where the total energy (averaged over the coordinate and time) is plotted as a function of γ, breaking of the PT-symmetry around $\gamma = 0.07$ can be observed. A similar effect can be seen from figure 6(b), where the average quasienergy is plotted as a function of γ. The critical value, at which PT-symmetry breaking occurs, is much smaller than that for PT-symmetric kicked rotors considered in [36]. This might be the result of the confinement in our system.

4. Conclusions

We studied quantum dynamics of a particle confined in a 1D box and driven by a PT-symmetric, delta-kicking potential. Different characteristics of the dynamics, such as the time-dependence of the average kinetic energy, quasienergy, and the average total energy have been analyzed using the exact solution of the time-dependent Schrödinger equation for a single kicking period. It is found that for the PT-symmetric case, the energy gain and acceleration are more rapid than those for the Hermitian counterpart, although no unbound acceleration is possible. Breaking of PT-symmetry, that is manifested in the fact that the average total energy and quasi-energy become complex starting from certain values of the non-Hermitian parameter, is found. The above model can be realized in optical systems with a PT-symmetric kicking force. The kick can be introduced using PT-symmetric lasers. One specific option could be, e.g. an optical cavity with losses and gains, where the standing wave is confined. Another option is considering a PT-symmetric periodic optical structure, e.g. arrays of optical waveguides driven by a PT-symmetric laser field. Such a system is described by the time-dependent Helmholtz equation with periodic boundary conditions, which is an analog of the box boundary condition.

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